IX. *A Dynamical Theory of the Electric and Luminiferous Medium.—* Part III. *Relations with Material Media.*

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1. In two previous memoirs[*] it has been explained, that the various hypotheses involved in the theory of electric and optical phenomena, which has been developed by FARADAY and MAXWELL, can be systematized by assuming the æther to be a continuous, homogeneous, and incompressible medium, endowed with inertia and with elasticity purely rotational. In this medium unitary electric charges, or electrons, exist as point-singularities, or centres of intrinsic strain, which can move about under their mutual actions ; while atoms of matter are in whole or in part aggregations of electrons in stable orbital motion. In particular, this scheme provides a consistent foundation for the electrodynamic laws, and agrees with the actual relations between radiation and moving matter.

An adequate theory of material phenomena is necessarily ultimately atomic. The older mathematical type of atomic theory which regards the atoms of matter as acting on each other from a distance by means of forces whose laws and relations are gradually evolved by observation and experiment, is in the present method expanded and elucidated by the introduction of a medium through whose intervention these actions between the material atoms take place. It is interesting to recall the circumstance that GAUSS in his electrodynamic speculations, which remained unpublished during his lifetime, arrives substantially at this point of view ; after examining a law of attraction, of the WEBERIAN type, between the "electric particles," he finally discards it and expresses his conviction, in a most remarkable letter to WEBER,[†] that

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[*] 'Phil. Trans.,' 1894, A, pp. 719–822 ; 1895, A, pp. 695–743 ; referred to subsequently as Part I. and Part II. [In the abstract of the present Memoir, 'Roy. Soc. Proc.,' 61, on p. 281, line 6, *read* $2\pi n'^2 + \int i' d\mathrm{F}$ for $2\pi n'^2$; line 35 *read* $\frac{2}{3} . \frac{4}{5}\pi i'^2$ for $\frac{4}{5}\pi i'^2$; and on p. 284, line 18, *read* $m/2c . \mathrm{E}\,(1 - m^2)$ for $\mathrm{E}\,(1 - m^2)$.]

[†] GAUSS, Werke, V., p. 629, letter to WEBER of date 1845 ; quoted by MAXWELL, "Treatise" II., § 861. After the present memoir had been practically completed, my attention was again directed, through a reference by ZEEMAN, to H. A. LORENTZ's Memoir "La Théorie Electromagnetique de MAXWELL et son application aux corps mouvants," Archives Néerlandaises 1892, in which (pp. 70 *seqq.*) ideas similar to the above are developed. The electrodynamic scheme at which he arrives is formulated differently from that given in § 13 *infra*, the chief difference being that in the expression for the electric force (P, Q, R) the term $- d/dt$ (F, G, H) is eliminated by introducing the æthereal displacement (f, g, h). This applies also to the later "Versuch einer Theorie . . . in bewegten Körpern,' 1895. The author

the key-stone of electrodynamics will be found in an action propagated in time from one "electric particle" to another. The abstract philosophical distinction between actions at a distance and contact actions, which dates for modern science from GILBERT'S adoption of the scholastic axiom* *Nulla actio fieri potest nisi per contactum*, can have on an atomic theory of matter no meaning other than in the present sense. The question is simply whether a wider and more consistent view of the actions between the molecules of matter is obtained when we picture them as transmitted by the elasticity and inertia of a medium by which the molecules are environed, or when we merely describe them as forces obeying definite laws. But this medium itself, as being entirely supersensual, we must refrain from attempting to analyse further. It would be possible (*cf.* § 6) even to ignore the existence of an æther altogether, and simply hold that actions are propagated in time and space from one molecule of matter to the surrounding ones in accordance with the system of mathematical equations which are usually associated with that medium; in strictness nothing could be urged against such a procedure, though, in the light of our familiarity with the transmission of stress and motion by elastic continuous material media such as the atmosphere, the idea of an æthereal medium supplies so overwhelmingly natural and powerful an analogy as for purposes of practical reason to demonstrate the existence of the æther. The aim of a theory of the æther is not the impossible one of setting down a system of properties in which everything that may hereafter be discovered in physics shall be virtually included, but rather the practical one of simplifying and grouping relations and of reconciling apparent discrepancies in existing knowledge.

2. It would be an unwarranted restriction to assume that the properties of the æther must be the same as belong to material media. The modes of transmission of

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remarks on the indirect manner in which dynamical equations had to be obtained, mainly on account of the absence of any notion as to the nature of the connexion between the stagnant æther and the molecules that are moving through it. "Dans le chemin qui nous a conduit à ces équations nous avons rencontré plus d'une difficulté sérieuse, et on sera probablement peu satisfait d'une théorie qui, loin de dévoiler le mécanisme des phénomènes, nous laisse tout au plus l'espoir de le découvrir un jour" (§ 91). In the following year (1893) similar general ideas were introduced by VON HELMHOLTZ, in his now well-known memoir on the electrical theory of optical dispersion, in which currents of conduction are included: but his argument is very difficult, and the results are in discrepancy with those of LORENTZ and the present writer in various respects in which the latter agree; moreover they are not consistent with the optical properties of moving material media. Both these discussions, of LORENTZ and of VON HELMHOLTZ, are in the main confined to electromotive phenomena: the treatment of the mechanical forces acting on matter in bulk would require for basis a theory of the mechanical relations of molecular media such as is developed in this paper. The results in the paper by ZEEMAN, above referred to, "On the Influence of Magnetism on the Light emitted by a substance," Verslagen Akad. Amsterdam, Nov. 28, 1896, have an important bearing on the view of the dynamical constitution of a molecule that has been advanced in these papers, and illustrated by calculation in an ideal simple case in Part I., §§ 114–8; *cf.* 'Roy. Soc. Proc.,' 60, 1897, p. 514. [See 'Phil. Mag.,' Dec. 1897: where the loss of energy by radiation from the moving ions is also examined.]

* GILBERT, *de Magnete*, 1600.

stress by media sensibly continuous were however originally formulated in connexion with the observed properties of elastic matter ; and the growth of general theories of stress-action was throughout checked and vivified by comparison with those properties. It was thus natural in the first instance to examine whether a restriction to the material type of elastic medium forms an obstacle in framing a theory of the æther ; but when that restriction has been found to offer insuperable difficulties it seems to be equally natural to discard it. Especially is this the case when the scheme of properties which specifies an available medium turns out to be intrinsically simpler than the one which specifies ordinary isotropic elastic matter treated as continuous.

A medium, in order to be available at all, must transmit actions across it in time ; therefore there must be postulated for it the property of inertia,—of the same kind as ordinary matter possesses, for there can hardly be a more general kind,—and also the property of elasticity or statical resistance to change either of position or of form. In ordinary matter the elasticity has reference solely to deformation ; while in the constitution here assumed for the æther there is perfect fluidity as regards form, but elastic resistance to rotational displacement.* This latter is in various ways formally the simpler scheme ; elasticity depending on rotation is geometrically simpler and more absolute than elasticity depending on change of shape ; and moreover no phenomenon has been discovered which would allow us to assume that the property of elasticity of volume, which necessarily exists in any molecularly constituted medium such as matter, is present in the æther at all. The objection that rotational elasticity postulates absolute directions in space need hardly have weight when it is considered that a definite space, or spacial framework fixed or moving, to which motion is referred, is a necessary part of any dynamical theory. The other fundamental query, whether such a scheme as the one here sketched could be consistent with itself, has perhaps been most convincingly removed by Lord KELVIN's actual specification of a gyrostatic cellular structure constituted of ordinary matter, which has to a large extent these very properties ; although the deduction of the whole scheme of relations from the single formula of Least Action, in its ordinary form in which the number of independent variables is not unnaturally increased, includes its ultimate logical justification in this respect.

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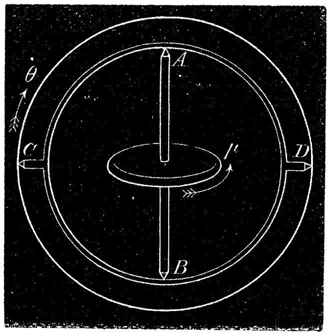
* I find that the rotational æther of MACCULLAGH, which was advanced by him in the form of an abstract dynamical system (for reasons similar to those that prompted MAXWELL to finally place his mechanism of the electric field on an abstract basis) was adopted by RANKINE in 1850, and expounded with full and clear realization of the elastic peculiarities of a rotational medium : by him also the important advantage for physical explanation, which arises from its fluid character, was first emphasized. *Cf.* Miscellaneous Scientific Papers, pp. 63, 160. In RANKINE's special and peculiar imagery, the æther was however a polar *medium* or *system* (as contrasted with a *body*) made up of polarized nuclei (*Cf.* Part I., §§ 37–8) whose vortical atmospheres, where such exist, constitute material atoms. The supposed necessity of having the vibration at right angles to the plane of polarization also misled him to the introduction of complications into the optical theory, such as æolotropic inertia, and to deviations from MACCULLAGH's rigorous scheme.

On Material Models and Illustrations of the Æther and its associated Electrons.

3. Although the GAUSSIAN aspect of the subject, which would simply assert that the primary atoms of matter exert actions on each other which are transmitted in time across space in accordance with MAXWELL'S equations, is a formally sufficient basis on which to construct physical theory, yet the question whether we can form a valid conception of a medium which is the seat of this transmission is of fundamental philosophical interest, quite independently of the fact that in default of the analogy at any rate of such a medium this theory would be too difficult for development. With a view to further assisting a judgment on this question, it is here proposed to describe a process by which a dynamical model of this medium can be theoretically built up out of ordinary matter,—not indeed a permanent model, but one which can be made to continue to represent the æther for any assignable finite time, though it must ultimately decay. The æther is a perfect fluid endowed with rotational elasticity; so in the first place we have—and this is the most difficult part of our undertaking—to construct a material model of a perfect fluid, which is a type of medium nowhere existing in the material world. Its characteristics are continuity of motion and absence of viscosity: on the other hand in an ordinary fluid, continuity of motion is secured by diffusion of momentum by the moving molecules, which is itself viscosity, so that it is only in motions such as vibrations and slight undulations where the other finite effects of viscosity are negligible, that we can treat an ordinary fluid as a perfect one. If we imagine an aggregation of frictionless solid spheres, each studded over symmetrically with a small number of frictionless spikes (say four) of length considerably less than the radius,* so that there are a very large number of spheres in the differential element of volume, we shall have a possible though very crude means of representation of an ideal perfect fluid. There is next to be imparted to each of these spheres the elastic property of resisting absolute rotation; and in this we follow the lines of Lord KELVIN'S gyrostatic vibratory æther. Consider a gyrostat consisting of a flywheel spinning with angular momentum μ, with its axis AB pivoted as a diameter on a ring whose perpendicular diameter CD is itself pivoted on the sphere, which may for example be a hollow shell with the flywheel pivoted in its interior; and examine the effect of imparting a small rotational displacement to the sphere. The direction of the axis of the gyrostat will be displaced only by that component of the rotation



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[*] The use of these studs is to maintain continuity of motion of the medium without the aid of viscosity; and also (§ 4) to compel each sphere to participate in the rotation of the element of volume of the medium, so that the latter shall be controlled by the gyrostatic torques of the spheres.

which is in the plane of the ring; an angular velocity $d\theta/dt$ in this plane will produce a torque measured by the rate of change of the angular momentum, and therefore by the parallelogram law equal to $\mu d\theta/dt$ turning the ring round the perpendicular axis CD, thus involving a rotation of the ring round that axis with angular acceleration $\mu/i \, . \, d\theta/dt$, that is with velocity $\mu/i \, . \, \theta$, where i is the aggregate moment of inertia of the ring and the flywheel about a diameter of the wheel. Thus when the sphere has turned through a small angle θ, the axis of the gyrostat will be turning out of the plane of θ with an angular velocity $\mu/i \, . \, \theta$, which will persist uniform so long as the displacement of the sphere is maintained. This angular velocity again involves, by the law of vector composition, a decrease of gyrostatic angular momentum round the axis of the ring at the rate $\mu^2/i \, . \, \theta$; accordingly the displacement θ imparted to the sphere originates a gyrostatic opposing torque, equal to $\mu^2/i \, . \, \theta$ so long as $\mu/i \, . \int \theta dt$ remains small, and therefore of purely elastic type. If then there are mounted on the sphere three such rings in mutually perpendicular planes, having equal free angular momenta associated with them, the sphere will resist absolute rotation in all directions with isotropic elasticity. But this result holds only so long as the total displacement of the axes of the flywheels is small: it suffices however to confer rotatory elasticity, as far as is required for the purpose of the transmission of vibrations of small displacement through a medium constituted of a flexible framework with such gyrostatic spheres attached to its links, which is Lord KELVIN'S gyrostatic model* of the luminiferous working of the æther. For the present purpose we require this quality of perfect rotational elasticity to be permanently maintained, whether the disturbance is vibratory or continuous. Now observe that if the above associated free angular momentum μ is taken to be very great, it will require a proportionately long time for a given torque to produce an assigned small angular displacement, and this time we can thus suppose prolonged as much as we please: observe further that the motion of our rotational æther in the previous papers is irrotational except where electric force exists which produces rotation proportional to its intensity, and that we have been compelled to assume a high coefficient of inertia of the medium, and therefore an extremely high elasticity in order to conserve the ascertained velocity of radiation, so that the very strongest electric forces correspond to only very slight rotational displacements of the medium: and it follows that the arrangement here described, though it cannot serve as a model of a field of steady electric force lasting for ever, can yet theoretically represent such a field lasting without sensible decay for any length of time that may be assigned.

4. It remains to attempt a model (*cf.* Part I., § 116) of the constitution of an electron, that is of one of the point-singularities in the uniform æther which are taken to be the basis of matter, and at any rate are the basis of its electrical phenomena. Consider the medium composed of studded gyrostatic spheres as above: although the motions of the æther, as distinct from the matter which flits across it,

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* Lord KELVIN, 'Comptes Rendus,' Sept. 1889: 'Math. and Phys. Papers,' III., p. 466.

are so excessively slow on account of its great inertia that viscosity might possibly in any case be neglected, yet it will not do to omit the studs and thus make the model like a model of a gas, for we require rotation of an individual sphere to be associated with rotation of the whole element of volume of the medium in which it occurs. Let then in the rotationally elastic medium a narrow tubular channel be formed, say for simplicity a straight channel AB of uniform section : suppose the walls of this channel to be grasped, and rotated round the axis of the tube, the rotation at each point being proportional for the straight tube to $AP^{-2} + PB^{-2}$: this rotation will be distributed through the medium, and as the result there will be lines of rotational displacement all starting from A and terminating at B : and so long as the walls of the channel are held in this position by extraneous force, A will be a positive electron in the medium, and B will be the complementary negative one. They will both disappear together when the walls of the channel are released. But now suppose that before this release the channel is filled up (except small vacuous nuclei at A and B which will assume the spherical form) with studded gyrostatic spheres so as to be continuous with the surrounding medium ; the effort of release in this surrounding medium will rotate these spheres slightly until they attain the state of equilibrium in which the rotational elasticity of the new part of the medium formed by their aggregate provides a balancing torque, and the conditions all round A or B will finally be symmetrical. We shall thus have created two permanent conjugate electrons A and B ; each of them can be moved about through the medium, but they will both persist until they are destroyed by an extraneous process the reverse of that by which they are formed. Such constraints as may be necessary to prevent division of their vacuous nuclei are outside our present scope ; and mutual destruction of two complementary electrons by direct impact is an occurrence of infinitely small probability. The model of an electron thus formed will persist for any finite assignable time if the distribution of gyrostatic momentum in the medium is sufficiently intense : but the constitution of our model of the medium itself of course prevents, in this respect also, absolute permanence. It is not by any means here suggested that this circumstance forms any basis for speculation as to whether matter is permanent, or will gradually fade away. The position that we are concerned in supporting is that the cosmical theory which is used in the present memoirs as a descriptive basis for ultimate physical discussions is a consistent and thinkable scheme ; one of the most convincing ways of testing the possibility of the existence of any hypothetical type of mechanism being the scrutiny of a specification for the actual construction of a model of it.

5. An idea of the nature and possibility of a self-locked intrinsic strain, such as that here described, may be facilitated by reference to the cognate example of a *material* wire welded into a ring after twist has been put into it. We can also have a closer parallel, as well as a contrast ; if breach of continuity is produced across an element of interface in the midst of an incompressible medium endowed with

ordinary material rigidity, for example by the creation of a lens-shaped cavity, and the material on one side of the breach is twisted round in its plane, and continuity is then restored by cementing the two sides together, a model of an electric doublet or polar molecule will be produced, the twist in the medium representing the electric displacement and being at a distance expressible as due to two conjugate poles in the ordinary manner. Such a doublet is permanent, as above ; it can be displaced into a different position, at any distance, as a strain-form, without the medium moving along with it ; such displacement is accompanied by an additional strain at each point in the medium, namely, that due to the doublet in its new position together with a negative doublet in the old one. A series of such doublets arranged transversely round a linear circuit will represent the integrated effect of an electric polarization-current in that circuit ; they will imply irrotational linear displacement of the medium round the circuit after the manner of vortex motion, but this will now involve elastic stress on account of the rigidity. Thus with an ordinary elastic solid medium, the phenomena of dielectrics, including wave-propagation, may be kinematically illustrated ; but we can thereby obtain no representation of a single isolated electric charge or of a current of conduction, and the laws of optical reflexion would be different from the actual ones. This material illustration will clearly extend to the dynamical laws of induction and electromagnetic attraction between alternating currents, but only in so far as they are derived from the kinetic energy ; the law of static attraction between doublets of this kind would be different from the actual electric law.

6. According to the present scheme the ponderomotive forces acting on matter arise from the forces acting on the electrons which it involves ; the application of the principle of virtual work to the expression for the strain-energy shows that, for each electron at rest, this force is equal to its charge multiplied by the intensity of the electric field where it is situated. It has been urged that a model of the æthereal electric field cannot be complete, and so must be rejected, unless it exhibits a direct mechanism by which the ponderomotive normal traction $F^2/8\pi$ is transmitted across the æther from the surface of one conducting region to that of another: but the position can be maintained that such a representation would transcend the limitations belonging to a mechanical model of a process which is in part mechanical and in part ultra-mechanical. Indeed if this force were transmitted in the ordinary elastic sense, the transmitting stress would have to be of the nature of a self-balancing FARADAY-MAXWELL stress involving the square of the æther-strain instead of its first power, and thus not directly related to elastic propagation. The model above described is so to speak made of æther, and ought to represent all the tractions that exist in æther, vanishing as they do over the surface of a conducting region : but the model does not in the ordinary sense represent matter at all, except in so far as the æthereal strain-form which constitutes the electron is associated with matter. It therefore cannot represent directly, after the manner of a stress across a medium, a force

acting on matter, for that would from this ultimate standpoint be a force acting on a strain-form spreading from its nucleus all through the medium, not a traction on a definite surface bounding the matter.

The fact is that transmission of force by a medium, or by contact action so-called, remains merely a vague phrase until the strain-properties of that medium are described; the scientific method of describing them is to assign the mathematical function which represents its energy of strain, and thence derive its relations of stress by the principle of virtual work; a real explanation of the transmission of a force by contact action must be taken to mean this process. Now in an elastic medium permeated by centres of permanent intrinsic strain, whether it be the rotational æther with its contained electrons, or an ordinary elastic solid permeated by polar strain-nuclei as described above, the specification of the strain-energy of the medium involves a mathematical function, not only of the displacement at each material point of the medium, but also of the positions of these intrinsic strain-centres which can move independently through it. To derive the play of internal force, this energy function must be varied with respect to all these independent quantities; the result is elastic tractional stress in the medium across every ideal interface, *together with* forcive tending to displace each strain-centre, which we can consider either as resisted by extraneous constraint preventing displacement of the strain-centre, or as compensated by the reaction of the inertia of the strain-form against acceleration.* Consider, for example, the analogy of the elastic solid medium, and suppose a portion of it to be slowly strained by extraneous force; two strains are thereby set up in it, namely that strain which would be thus originated if the solid were initially devoid of intrinsic strain, and that strain which has to be superposed in order to attain the new configuration of the intrinsic strain arising from the displacement of its nuclei. The latter part is conditioned by the displacement of these strain-centres, and in its production forces acting on them must be considered to assist, whose intensities may be determined as has been already done in the æthereal problem.

The attractions between material bodies are therefore not transmitted by the æther in the way that mechanical tractions are transmitted by an ordinary solid, for it is electric force that is so transmitted: but neither are they direct actions at a distance. The point of view has been enlarged: the ordinary notion of the transmission of force, as framed mathematically by LAGRANGE and GREEN for a simple elastic medium without singularities, is not wide enough to cover the phenomena of a medium containing intrinsic strain-centres which can move about independently of the substance of the medium. But the same mathematical principles lead to the necessary extension of the theory, when the energy function thus involves the positions of the

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* Thus when the medium is in equilibrium, there is in it only the static intrinsic strain diverging from these centres, which gives rise to the forces between them; but when it is disturbed by radiation or otherwise, there is also the strain thence arising.

strain-centres as well as the elastic displacement in the medium; and the theory which in the simpler case answers fairly to the description of transmission by contact action has features in the wider case to which that name does not so suitably apply.* The strain-centres (that is, the matter) have, in the strict sense of the term, *energy of position*, or *potential energy*, due to their mutual configuration in the æther, which can come out as work done by mutual forces between them when that configuration is altered, which work may be used up either in accumulating other potential energy elsewhere, or in increasing the kinetic energy of the matter, which is itself, in whole or in part, energy in the æther arising from the movement of the strain-forms across it. Discussions as to transmission by contact are not the fundamental ones, as the above actual material illustration shows: the single comprehensive basis of dynamics into which all such partial modes of explanation and representation must fit and be coordinated is the formula of Stationary Action, including, as the particular case which covers all the domain of steady systems, the law that the mutual forces of such a system are derived from a single analytical function which is its available potential energy.

The circumstance that no mode of transmission of the mechanical forces, of the type of ordinary stress across the æther, can be put in evidence, thus does not derogate from the sufficiency of the present standpoint. The transmission of material traction by an ordinary solid, which is now often taken as the type to which all physical action must conform, is merely an undeveloped notion arising from experience, which must itself be analysed before it becomes of scientific value: the explanation thereof is the quantitative development of the notion from the energy function by the method of virtual work in the manner indicated in § 10 *infra*. This orderly development of the laws of action across a distance, from an analytical specification of a distribution of energy pervading the surrounding space, is the essence of the so-called principle of contact action. It is precisely what the present procedure carries out, with such generalization as the scope of the problem demands; besides attaining a correlation of the whole range of the phenomena, it avoids the antinomies of partial theories which accumulate on the æther contradictory and unrelated properties, and sometimes even save appearances by passing on to the simple fundamental medium those complex properties of viscous matter whose real origin is to be found in its molecular discreteness.

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* An analogous principle applies in the vortex-theory illustration of matter. If we consider rigid cores round which the fluid circulates, they are moved about by the fluid pressure: but if we consider vortex-rings, say with vacuous cores, these are mere forms of motion that move across the fluid, and if we take them to represent atoms, the interactions between aggregations of atoms cannot be traced by means of fluid pressures, but can only be derived from the analytical character of the function which expresses the energy.

Æther contrasted with Matter.

7. The order of development here followed is thus avowedly based on the hypothesis that the æther is a very simple uniform medium, about which it may be possible to know all that concerns us; and the present state of the theories of optics and electricity does much to encourage that idea. This procedure is of course at variance with the extreme application of the inductive canon, which would not allow the introduction of any hypothesis not based on direct observation and experiment. But though that philosophy has abundantly vindicated itself as regards the secondary properties of matter, which are amenable to direct examination, its rigid application would debar us from any theory of the æther at all, as we can only learn about it from circumstantial evidence. We could then merely go on heaping up properties on the æther, on the analogy of what is known of matter, as circumstances necessitated; and this medium would be a sort of sink to dispose of relations that could not be otherwise explained. Whereas matter, with which we are familiar, is the really complicated thing on which all the maze of physical phenomena depends, so that it is doubtful whether much can ever be known definitely as to its ultimate dynamical constitution; our best chance is to try to approach it through the presumably simple and homogeneous æther in which it subsists.

For example, it is found that the transmission of electrostatic force is affected by the constitution of the material dielectric through which it passes, and this is explained by a perfectly valid theory of polarization of the molecules of the matter: to press the analogy and ascribe the possibility of transmission through a vacuum to polarization of the æther may be convenient for some purposes of description, but in the majority of cases the impression is left that the so-called polarization of the æther is thereby explained. Whereas the processes being, almost certainly, of totally different character in the two cases, it will conduce to accurate thought to altogether avoid using the same term in the two senses, and to speak of the *displacement* of the æther which transmits electric force across a vacuum as producing *polarization* in the molecules of a material dielectric which exists in its path, which latter in turn affects the transmission of the electric force by reaction. In trying to pass beyond this stage, we may accumulate descriptive schemes of equations, which express, it may be with continually increasing accuracy, the empirical relations between these two phenomena; but we can never reach very far below the surface without the aid of simple dynamical working hypotheses, more or less *a priori*, as to how this interaction between continuous æther and molecular matter takes place.

8. On the present view, physical theory divides itself into two regions, but with a wide borderland common to both: the theory of radiation or the kinetic relations of this ultimate medium; and the theory of the forces of matter which deals for the most part with molecular movements so slow that the surrounding æther is at each instant practically in an equilibrium condition, so that the material atoms practically

act on each other from a distance with forcives obeying definite laws, derivable from the formula for the energy. It is only in electromagnetic phenomena and molecular theory that non-vibrational movements of the æther are involved. The æther not being matter, it need not obey the laws of the dynamics of matter, provided it obey another scheme of dynamical laws consistent among themselves; these laws must however be such that we can construct in the æther an atomic system of matter which itself obeys the actual material laws. The sole spacial relations of the æther itself, on which its dynamics depend, those namely of incompressibility and rotational elasticity, are thus to be classed along with the existing EUCLIDEAN relations of measurements in space (which also might *a priori* be different from what they are) as part of the ultimate scheme of mental representation of the actual physical world. The elastic and other characteristics of ordinary matter, including its viscous relations, are on the other hand a direct consequence of its molecular constitution, in combination with the law of material energy which is itself a consequence of the fact that the energies of the atoms are wholly located in the surrounding simple continuous æther and are thus functions of their mutual configurations. In this way we come round again to an order of procedure similar to that by which CAUCHY and POISSON originally based the elastic relations of material bodies on the mutual actions of their constituent molecules.

Consider any two portions of matter which have a potential-energy function depending, as above explained, on their mutual configuration alone, the material movements being thus comparatively slow compared with the velocity of radiation; any displacement of them as a single rigid system, whether translational or rotational, can involve no expenditure of work; hence the resultant forcive exerted by the first system on the second must statically equilibrate that exerted by the second system on the first, these forcives must in fact be equal and opposite wrenches on a common axis; and the energy principle thus involves the principle of the balance of action and reaction, in its most general form. This stress, between two molecules, is usually sensible only at molecular range; hence the action of the surrounding parts on a portion of a solid body is practically made up of tractions exerted over the interface between them. Further, since rotation of the body without deformation cannot alter the potential energy of mutual configuration of the molecules, it follows that for a rectangular element of ordinary solid matter the tangential components of these tractions must be self-conjugate, as they are taken to be in the ordinary theory of elasticity. On the other hand, for a medium not molecularly constituted we can hardly treat at all of mutual configuration of parts, and the self-conjugate stress-relation will not be a necessary one.

A certain similarity may be traced with the view of FARADAY, who was disinclined to allow that ray-vibrations are transmitted by any medium of the molecular character of ordinary matter, but considered them rather as affections of the lines which represent electric force, the propagation being influenced by the material nuclei which in ponderable media disturb these lines. This propagation in time requires inertia and elasticity for its mathematical expression, and the problem of the free æther is to find what kind of each is requisite.

9. A theory which, like the present one, explains atoms of matter as made up of singularities of strain and motion in the æther, is bound to look for an explanation of gravitation by means of the properties of that medium ; it cannot avail itself of COTES's dogma that gravitation at a distance is itself as fundamental and intelligible as any explanation thereof could be. In further development of the illustrative possibilities of the pulsatory theory of gravitation, mentioned in the previous papers, we can (ideally) imagine the pulsation to have been applied initially over the outside boundary of the æthereal universe, and thence instantaneously communicated throughout the incompressible medium to the only places that can respond to it, the vacuous nuclei of the electrons ; and we can even imagine the pulsations thus established as spontaneously keeping time and phase ever after, when the exciting cause which established this harmony has been discontinued.

It has been noticed in Part I, § 103, that gravitation cannot be transmitted by any action of the nature of statical stress ; for then the approach of two atoms would increase the strain, and therefore also the stress, and therefore also in a higher ratio the energy of strain which depends on their product, and hence the mutual forces of the atoms would resist approach. As gravitation must belong to the ultimate constituents of matter, that is on this theory to the electrons, and must be isotropic all round each of them, it would appear that no mediate æthereal representation of it is possible except the one here considered. The radially vibrating field might be described formally as the magnetic field of the electron considered as a unipolar magnet, necessarily of very rapidly alternating type because otherwise a field of gravitation would be an ordinary magnetic field. The bare groundwork of this hypothesis may thus be formally expressed in MAXWELL's language and developed along his lines, by postulating that the electron is not only a centre of steady intrinsic electric force, but also a centre of alternating intrinsic magnetic force, instantaneously transmitted because it would otherwise involve condensation, each force being necessarily radial.* The unsatisfactory feature is that this radial *quasi*-magnetic field is introduced for the sake of gravitation alone, which does not present itself as in any direct correlation with other physical agencies.

The following sections are occupied chiefly with an attempt to logically systematize, and in various respects extend, the electric aspect of molecular theory. The preceding paper dealt mainly with the molecular side of directly æthereal phenomena, such as electric and radiative fields ; of the present one the earlier part follows up the same subject, and the remainder relates to the actions of the molecules of polarized

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* Two steady magnetic poles of like sign would repel each other : but in the case of two poles pulsating in the same phases there is also an inertia term in the fluid æthereal pressure, and the result is as stated above. *Cf.* HICKS, 'Proc. Camb. Phil. Soc.,' 1880, p. 35.

material bodies on one another, and the material stresses and physical changes thereby produced. As in the preceding papers, the quantitative results are to a large extent independent of any special theory of the constitution of matter, such as is here employed to bind together and harmonize the separate groups of phenomena, and to form a mental picture of their mutual relations; so far as they are electric they may be based directly on MAXWELL'S equations of the electric field in *free space*, which form a sufficient description of the free æther, and have been verified by experiment. In the FARADAY-MAXWELL theory, however, as usually expounded, an explanation of these equations is found, explicitly or tacitly, in an assumption that the æther is itself polarizable in the same manner as a material medium, and æther is in fact virtually considered to be matter; on the present theory the equations for free space are an analytical statement of the ultimate dynamical definition of the continuous æthereal medium, and the polarization of material bodies with the resulting forcive are deduced from the relation of their molecules to this medium in which they have their being.

10. In the modern treatment of material dynamics, as based on the principle of energy, the notion of configuration is, as above explained, fundamental. The potential energy, from which the forces are derived, is a function of the mutual configurations of the parts of the material system. In the case of forces of elasticity the internal energy is primarily a function of the mutual configurations of the individual molecules, from which a regular or organised part (§ 49 *infra*) is separated which is expressible in terms of the change of configuration of the differential element of volume containing a great number of molecules, and from which alone is derived the stress that is mechanically transmitted. In connexion with the discussion of contact action in § 6 above, the mode of this derivation and transmission becomes a subject of interest.* In the first place the primary notion of a *force* as acting from one point to another in a straight line, has to be generalized into a *forcive* in LAGRANGE'S manner on the basis of the principle of virtual work: then the forcive arising from the internal strain-energy of the element of volume of the material is derived by variation of this organized energy, and appears primarily as made up of definite complex bodily forcives resisting the various types of strain that occur in the element: then these forcives are rearranged, by the process of integration by parts, into a uniform translatory force acting throughout the element of volume of the material (which must compensate the extraneous applied bodily forcive) together with tractions acting over its surface. When this is done also for adjacent elements of volume, other tractions arise which must compensate the

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* It is here assumed that the direct action between the molecules is sensible only at molecular distances, which would not be the case if the material were electrically polarized. The statement also refers solely to transmitted mechanical stress of the ordinary kind: more complicated types, not expressible by surface tractions alone, are put aside, as well as molecular conceptions like the LAPLACIAN intrinsic pressure in fluids. *Cf.* §§ 44–6 *infra*.

previous ones over the part of the surface that is common to the two elements; and thus the uncompensated traction is passed on from element to element until finally the boundary of the material system is reached where it remains uncompensated and must be balanced extraneously. The outstanding irregular part of the aggregate mutual potential energy of the individual molecules, which cannot be included in a function of strain of the element of volume, cannot on that account take part in the transmission of mechanical forces, and is evidenced only in local changes of the physical properties and temperature of the material. *Cf.* § 48 *infra.*

The other main division of the energy is the kinetic part, which is specified in terms of the rate of change of configuration of the material system with respect to an extraneous spacial framework to which its position is referred. Whatever notions may commend themselves *a priori* as to the impossibility of absolute space and absolute time, the fact remains that it has not been found possible to construct a system of dynamics which has respect only to the relative positions of moving bodies; and the reason suggests itself, that there is an underlying part of the phenomena, which does not usually explicitly appear in abstract material dynamics, namely, the æthereal medium, and that the spacial framework in absolute rest, which was introduced by NEWTON and was probably a main source of the great advance in abstract dynamics originated by the *Principia,* is in fact the quiescent underlying æther. In this way the purely *a priori* standpoint is pushed away a stage, and we may find justification against the reproach that a philosophical formulation of dynamics should be concerned only with relative motions.

Relation to Gas-Theory: Internal Molecular Energy.

11. The kinetic theory of gases is considerably affected by the view here taken of the constitution of a molecule. In those simple and satisfactory features which are concerned only with the translatory motion of the molecules, it stands intact; but it is different with problems, like that of the ratio of the specific heats, which involve the internal energy. According to the usual hypothesis of the theory of gases, all the internal kinetic energy of the molecule is taken to be thermal and in statistical equilibrium, through encounters, with the translatory energy. But on the present view, the energy of the steady orbital motions in the molecule (including therein slow free precessions) makes up both the energy of chemical constitution and the internal thermal energy; while it is only when these steady motions are disturbed that the resulting vibration gives rise to radiation by which some of the internal energy is lost. The amount of internal energy can however never fall below the minimum that corresponds to the actual conserved rotational momenta of the molecule; this minimum is the energy of chemical combination of its ultimate constituents, while the excess above it actually existing is the internal thermal energy.* The present

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* As a concrete illustration, we can imagine two ideal atoms, each consisting of a single gyrostat

view requires that the energy of chemical constitution shall be very great compared with the thermal energy; but for this very reason our means of chemical decomposition are limited, so that only a part of that energy is experimentally realizable.* This being the case, the alteration produced by external disturbance in the state of steady internal motions of the molecule consists in the superposition on it of very slow free precessional motions, which have practically no influence on its higher free periods:† and this explains why change of temperature has no influence on the positions of the lines in a spectrum. As a gas at high temperature must contain molecules with all amounts of internal thermal energy from nothing upwards, we should on the other hand, on the ordinary gas-theory, expect both a shift of the brightest part of a spectral line when the temperature is raised, and also a widening of its diffuse margin.

The ordinary encounters between the molecules will influence this thermal energy or energy of slow precessional oscillation, without disturbing the state of steady constitutive motion on which it is superposed, therefore without exciting radiation, which depends on more violent disturbances involving dissociative action.

On this view the postulates of the MAXWELL-BOLTZMANN theorem on the distribution of the internal energy in gases would not obtain, for the thermal energy of the molecule would not be expressible as a sum of squares. The ratio of the specific heats in a gas must still lie between 1 and $\frac{5}{3}$; but the nature of the similarity of molecular constitution in the more permanent gases, which makes the ratio of the total thermal energy to the translatory energy either $\frac{5}{3}$ or unity for most of them, would remain to be discovered. In those gases for which the latter value obtains, the energy of precessional motion in the molecule would be negligibly small, involving small resultant angular momentum and *possibly* small paramagnetic moment.

The necessity of a distinction such as that here drawn between the internal thermal energy and the energy of the vibratory disturbances of internal structure which maintain radiation, is well illustrated by the recent recognition (foreshadowed by DULONG and PETIT's researches on the law of cooling) and application by DEWAR of the remarkable insulating power of a vacuum jacket as regards heat. If this distinction did not exist, both conduction and convection must ultimately depend on

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enclosed in a suitable massless case, coming into mutual encounter. We may imagine that neither of them has any internal heat; so that the internal energy of each is the minimum that corresponds to its steady gyrostatic momentum, and the axis of each gyrostat therefore keeps a fixed direction in space. The result of the encounter will be that the axis of each gyrostat acquires steady wobbling or free precessional motion, so that its internal energy is increased at the expense of the energy of translation of the atoms; but in this the simplest case there will be no unsteady vibration, such as could be radiated away. If however there are also other types of momenta associated with the atom, for example if the case of the gyrostat is not massless, the encounter will leave vibrations about the new state of steady motion, which if of high enough period will lead to loss of energy by radiation.

* Ideas somewhat similar to the above are advanced by WATERSTON in his classical memoir of 1845 on gas-theory, recently edited by Lord RAYLEIGH; 'Phil. Trans.' (A), 1892, p. 51.

† *Cf.* THOMSON and TAIT, 'Nat. Phil.' § 345 xxiv.

transfer by ordinary radiation at small distances, as FOURIER imagined; and it would not appear why convection by a gas, even when highly rarefied, is so much more efficient in the transfer of heat than radiation.

12. The result obtained by RAMSAY and YOUNG, and others, that all over the gas-liquid range the characteristic equations of the substances on which they experimented proved to be very approximately of the form $p = a\mathrm{T} + b$, where a and b are functions of the density alone, also supplies corroboration to this view. Expressing the increment of energy per unit mass $d\mathrm{E} = \mathrm{M}dv + \kappa d\mathrm{T}$, we have for the increment of heat supplied $d\mathrm{H} = d\mathrm{E} + pdv$; and the fact that $d\mathrm{E}$ and $d\mathrm{H}/\mathrm{T}$ are perfect differentials shows immediately that M is equal to $-b$ and κ is independent of v, so that the total (non-constitutive) energy per unit mass consists of two independent parts, an energy of expansion and an energy of heating.* The latter part is the thermal energy of the individual molecules; it is a function of their mean states and velocities alone, and constitutes almost all the energy in the gaseous state. The former part is the energy of mutual actions between the molecules; it is negative and bears a considerable ratio to the whole thermal energy in the liquid state, in the case of substances with high latent heats of evaporation; for all gases except hydrogen, inasmuch as they are cooled by transpiration through a porous plug, b is negative at ordinary densities. *Cf.* § 62, *infra.*

There would be no warrant for a view that the forces of chemical affinity fall off and finally vanish as the ultimate zero of temperature is approached. The translatory motions of the molecules would diminish without limit, and therefore also the opportunities for reaction between them, so that many chemical changes would cease to take place for the same reason that a fire ceases to burn when the supply of air is insufficient, or coal gas ceases to explode when too much diluted with air: but the energies of affinity exist all the time in probably undiminished strength, while the forces of cohesion are modified by the fall of temperature but not necessarily in the direction of extinction.

The Equations of the Æthereal Field, with Moving Matter: various applications: influence of Motion through the Æther on the Dimensions of Bodies.

13. Let (u, v, w) represent the total circuital current, and (u', v', w') the conducted part of it, which will be taken to include the current (u_0, v_0, w_0) of migration of the free electric charge as this is in all cases very small in comparison; let (f', g', h') denote the electric polarization of the material, and (f, g, h) the æthereal elastic displacement, so that the total circuital displacement of MAXWELL's theory is their sum (f'', g'', h''); let the space of reference be fixed with respect to the stagnant æther, and (p, q, r) be the velocity with which the matter situated at the point (x, y, z) is moving, and let δ/dt represent $d/dt + pd/dx + qd/dy + rd/dz$; let

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* *Cf.* G. F. FITZGERALD, 'Roy. Soc. Proc.' 42, 1887: *cf.* also CLAUSIUS' early ideas on 'disgregation.'

(P, Q, R) denote the electric force, namely that which acts on the electrons, and (P′, Q′, R′) the æthereal force, that which produces the æthereal electric displacement (f, g, h); let ρ denote density of free electric charge. Then the electromotive equations are*

$$\mathrm{P} = qc - rb - \frac{d\mathrm{F}}{dt} - \frac{d\Psi}{dx}, \quad \mathrm{P}' = -\frac{d\mathrm{F}}{dt} - \frac{d\Psi}{dx}, \quad f = \frac{1}{4\pi\mathrm{C}^2}\mathrm{P}',$$

where

$$\mathrm{F} = \int \frac{u}{r}\, d\tau + \int \left(\mathrm{B}\frac{d}{dz} - \mathrm{C}\frac{d}{dy}\right)\frac{1}{r}\, d\tau, \quad a = \frac{d\mathrm{H}}{dy} - \frac{d\mathrm{G}}{dz};$$

and

$$u = u' + \frac{\delta f'}{dt} + \frac{df}{dt} + p\rho,^{\dagger}$$

where

$$\rho = \frac{d(f'+f)}{dx} + \frac{d(g'+g)}{dy} + \frac{d(h'+h)}{dz}.$$

From the formula for (P, Q, R) FARADAY's law follows that the line integral of electric force round a circuit *in uniform motion with the matter* is equal to the time-rate of diminution of the magnetic flux through its aperture. The line-integral of the

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* This scheme forms an improved summary of that worked out in Part II. §§ 15–19; the expressions there assigned for ρ and Ψ have here been corrected, and (u_0, v_0, w_0) is merged.

† [*Added Sept.* 14.—The term $\delta f'/dt$ in u arises as follows. In addition to the change of the polarization in the element of volume, df/dt, there is the electrodynamic effect of the motion of the positive and negative electrons of the polar molecule. Now the movement of two connected positive and negative electrons is equivalent to that of a single positive electron round the circuit formed by joining together the ends of their paths: and a similar statement holds when there are more than two electrons in the molecule. Hence the motion of a polarized medium with velocity (p, q, r), which need not be constant from point to point, produces the electrodynamic effect of a magnetization $(rg' - qh', ph' - rf', qf' - pg')$ distributed throughout the volume: *cf.* Part I, § 125. And it has been shown in Part II, § 31 that any distribution of magnetism (A, B, C) may be represented as a volume distribution of electric current equal to curl (A, B, C), which is necessarily circuital, together with a surface current sheet equal to $(\mathrm{B}n - \mathrm{C}m, \mathrm{C}l - \mathrm{A}n, \mathrm{A}m - \mathrm{B}l)$. Thus, *when* (p, q, r) *is uniform and* (f', g', h') *is circuital*, the above magnetic distribution is equivalent to a current system $(pd/dx + qd/dy + rd/dz)\,(f', g', h')$ together with current sheets on interfaces of discontinuity: this system is to be added on to $d/dt\,(f', g', h')$ in order to give the full electrodynamic effect. Thus in these special circumstances the formulation in the test is correct in so far as it leads to the correct *differential* equations for the element of the medium: the *integral* expression there given for F is however only correct either when it is reduced to the differential form $-\nabla^2\mathrm{F}/4\pi = u + d\mathrm{C}/dy - d\mathrm{B}/dz$, which is derivable on integration of its second term by parts, or else when, the velocity of the matter still being uniform, discontinuous interfaces are replaced in the analysis by gradual though rapid transitions. These conditions are satisfied in all the applications that follow: but they would not be satisfied for example in the problem of the reflexion of radiation from the surface of moving matter.

But a formulation which is preferable to the above, in that it is *absolutely general*, is simply to implicitly include the above virtual magnetization directly in (A, B, C) and consequently change from $\delta f'/dt$ to df'/dt in the expression for u: this will also involve that the relation $\mathrm{A} = \kappa\alpha$ which occurs lower down shall be replaced by $\mathrm{A} = \kappa\alpha + rg' - qh'$, but there will be no further alteration in the argument of the text.. The boundary conditions of the text are unaltered.]

æthereal force (P′, Q′, R′) round a circuit *fixed in the æther* has the same value. Again if (F′, G′, H′) be defined so that $\mathrm{F}' = \int u/r \,.\, d\tau$, we have

$$a = \frac{d\mathrm{H}'}{dy} - \frac{d\mathrm{G}'}{dz} + 4\pi\mathrm{A} - \frac{d}{dx}\int\left(\mathrm{A}\frac{d}{dx} + \mathrm{B}\frac{d}{dy} + \mathrm{C}\frac{d}{dz}\right)\frac{1}{r}d\tau,$$

so that

$$\alpha + \frac{d\mathrm{V}'}{dx} = \frac{d\mathrm{H}'}{dy} - \frac{d\mathrm{G}'}{dz},$$

where (α, β, γ) is magnetic force and V′ is the potential of the magnetism: hence AMPÈRE's law follows that the line integral of the magnetic force round *any* circuit is equal to 4π times the total current that flows through its aperture. These two circuital relations are coextensive with the previous equations involving the vector potential, and can thus replace them, when the difference between (P, Q, R) and (P′, Q′, R′) is inessential, that is (i) when the displacement currents are negligible, or (ii) when the matter is at rest; the quantity Ψ then enters as an arbitrary function in the integration of the equations.

The mechanical force acting on the matter, or ponderomotive force, is (X, Y, Z) per unit volume, where (§ 38 *infra*)

$$\mathrm{X} = \left(v - \frac{dg}{dt}\right)\gamma - \left(w - \frac{dh}{dt}\right)\beta + \mathrm{A}\frac{d\alpha}{dx} + \mathrm{B}\frac{d\alpha}{dy} + \mathrm{C}\frac{d\alpha}{dz} + f'\frac{d\mathrm{P}}{dx} + g'\frac{d\mathrm{P}}{dy} + h'\frac{d\mathrm{P}}{dz} + \rho\mathrm{P}.$$

The mechanical traction on an interface will be considered later (§ 39). In a magnetic medium the magnetic force (α, β, γ) differs from the magnetic flux (a, b, c) simply by not including the influence of the local AMPEREAN currents; thus $\alpha = a - 4\pi\mathrm{A}$.

When there is no conductivity, the free charge must move along with the matter, so that

$$\frac{d\rho}{dt} + \frac{d\rho p}{dx} + \frac{d\rho q}{dy} + \frac{d\rho r}{dz} = 0;$$

therefore, from the circuitality of the total current, we must have, identically,

$$\frac{d\rho}{dt} = \frac{d}{dx}\left(\frac{\delta f'}{dt} + \frac{df}{dt}\right) + \frac{d}{dy}\left(\frac{\delta g'}{dt} + \frac{dg}{dt}\right) + \frac{d}{dz}\left(\frac{\delta h'}{dt} + \frac{dh}{dt}\right).$$

The latter is the same as the convergence of $(\delta/dt - d/dt)\,(f', g', h')$, which asserts (for the case of uniform motion that is contemplated) that mere convection of the polarized medium does not produce separation of free electricity. The relation between (f', g', h') and (P, Q, R) must be such as to strictly satisfy this equation. The quantity Ψ occurs in the equations of the field as an undetermined potential which is sufficient in order to conserve the condition of bodily circuitality $du/dx + dv/dy + dw/dz = 0$.

In order to express the conditions that must hold at an interface of transition, we notice that by definition F, G, H are continuous everywhere; but it is only when the media are non-magnetic that their rates of change along the normal (and therefore all their first differential coefficients) are also completely continuous. Across an interface the traction in the æther must be continuous, so that the tangential component of the æthereal force (P', Q', R') must be continuous, which is satisfied by continuity of Ψ. The continuity of the total electric current secures itself without further condition by a compensating distribution of electric charge on the interface, that is by a discontinuity in $d\Psi/dn$. The tangential continuity of the elastic æther requires that the tangential component of the magnetic force (α, β, γ) must be continuous; the normal continuity of the magnetic flux is assured by the continuity of (F, G, H). It might be argued that if the electric force (P, Q, R) were not continuous tangentially, a perpetual motion could arise by moving an electron along one side of the interface and back again along the other side. But this reasoning requires that (p, q, r) shall be continuous across the interface, as otherwise the circuit returning on the other side could not be complete; and it also requires that there shall be no magnetization, as otherwise the mechanical force on the electrons in an element of volume, which is what we are really concerned with in the perpetual motion axiom, is different from the sum of the electric forces on the individual electrons, by involving (α, β, γ) instead of (a, b, c). We can thus assert continuity of the tangential electric force only in the cases in which it is already involved in that of the tangential æthereal force; and consistency is verified. The aggregate of all these interfacial electromotive conditions is thus continuity of the vector potential (F, G, H), and of Ψ, and of the tangential components of the magnetic force; they *formally* involve continuity of the tangential components of the æthereal force (P', Q', R'), and of the electric and magnetic fluxes. But further, in the equations from which AMPÈRE's circuital relation is derived above, it is only the normal space-variation of V' that is discontinuous; hence continuity of the tangential magnetic force is involved in that of F, G, H, Ψ by virtue of the mode of expression of (F, G, H) in terms of the currents and the magnetism. Thus there are in all cases only four independent interfacial conditions to be satisfied.

The scheme is thus far absolute, in the sense that the relations between the variables are independent of the special molecular constitution of the matter that is present. The system of equations must now be completed for material media by joining to it the relations which connect the conduction current in the matter with the electric force, and the electric polarization of the matter with the electric force, and the magnetic polarization of the matter with the magnetic force, in the cases in which these relations are definite and can be experimentally ascertained. In the simplest case of isotropic matter, polarizable according to a linear law, they are of types

$$u' = \sigma P, \quad f' = (K - 1)/4\pi C^2 \, P, \quad A = \kappa\alpha.$$

The expression for ρ leads in homogeneous isotropic media to

$$K\nabla^2\Psi = - 4\pi C^2\rho + (K - 1) \{d/dx\,(cq - br) + d/dy\,(ar - cp) + d/dz\,(bp - aq)\}$$

so that Ψ is only in part an electrostatic potential. Inside a uniform isotropic conductor *at rest*, the condition of circuitality becomes $\sigma\nabla^2\Psi = d\rho/dt$; substituting this, we have $d\rho/dt + 4\pi C^2\sigma K^{-1}\rho = 0$, so that $\rho = \rho_0 \exp(-4\pi C^2\sigma K^{-1}t)$, showing that an initial volume density of free electricity would in that case be instantly driven to the boundary owing to the dielectric action. This proposition may be extended to æolotropic media.

14. The nature of the foregoing electric scheme may be elucidated by aid of some simple applications.

(i.) When a conducting system is in steady motion so that there is no *conduction* current flowing into it, the electric force (P, Q, R) must be null throughout its substance. Thus for the case of a solid conductor rotating round an axis of symmetry in a uniform magnetic field parallel to that axis, with steady angular velocity ω, the electric force in it, namely $(\omega cx - d\Psi_1/dx,\ \omega cy - d\Psi_1/dy,\ - d\Psi_1/dz)$, must be null, so that $\Psi_1 = \frac{1}{2}\omega c(x^2 + y^2) + A$; the polarization in it is therefore null, but there is in it an æthereal displacement $-(4\pi C^2)^{-1}(d/dx,\ d/dy,\ d/dz)\Psi_1$. In outside space, the electric force and æthereal force are each $-(d/dx,\ d/dy,\ d/dz)\Psi_2$, where Ψ_2 is that free electrostatic potential which is continuous with the surface value $\frac{1}{2}\omega c(x^2 + y^2) + A$ at the conductor. Inside the conductor this purely æthereal displacement involves an electrification of volume density $\rho = -\omega c/2\pi C^2$, which will be a density of free electrons or ions as all true electrifications are; while there is a compensating surface density σ equal to the difference of the total normal electric displacements on the two sides, that is to $(4\pi C^2)^{-1}(d\Psi_2/dn_2 + d\Psi_1/dn_1)$, where dn_2, dn_1 are both measured towards the surface, the outside medium being air for which K is unity. The value of the constant A is determined by the circumstance that the aggregate of this volume and surface charge shall be null when the conductor is insulated and unelectrified, or equal to the given total charge when it is insulated and charged: when it is uninsulated, the constant is determined by the position of the point on it that is connected to Earth, and therefore at zero potential. The procedure of Part II., § 25 is thus justified, because there is in fact no dielectric polarization in the conductor, but only æthereal displacement.

It remains to consider whether the parts of this volume density ρ and surface density σ of electrification are carried round with the conductor in its motion, or slip back through its volume and over its surface so as to maintain fixed positions in space. It is clear (as in Part II., § 27) that the same cause, namely, viscous diffusion of momentum among moving ions and molecules, which produces OHMIC resistance to a steady current, will lead to the electrons constituting electric densities being wholly carried on by the matter whenever a steady state is attained. This necessary consequence of the theory is in keeping with ROWLAND's classical experiments on convection currents. The excessively minute magnetic field due to these convection currents themselves has been neglected in the above analysis, which has enabled us to specify the slight redistribution of free charge on the rotating conductor when under the influence of a powerful extraneous magnetic field: when the magnetic field is due solely to its own motion the redistribution is of course absolutely negligible.

(ii.) In the case of a dielectric (as also in the above) the restriction to a steady state and permanent configuration may be dispensed with; for the magnetic field arising from induced displacement currents can always be neglected in comparison with the inducing field. Thus, (a, b, c) being the extraneous inducing field, the electric forces inside and outside a rotating mass are

$$(\omega cx - d\Psi_1/dx,\ \omega cy - d\Psi_1/dy,\ -\omega ax - \omega by - d\Psi_1/dz) \text{ and } -(d/dx,\ d/dy,\ d/dz)\,\Psi_2.$$

As there can be no free electrification,

$$\nabla^2\Psi_1 = (1 - \mathrm{K}^{-1})\,\omega\,\{2c + x(dc/dx - da/dz) + y(dc/dy - db/dz)\} \text{ and } \nabla^2\Psi_2 = 0\,;$$

while at the surface

$$\Psi_1 = \Psi_2, \text{ and } \mathrm{K}\,d\Psi_1/dn - (\mathrm{K} - 1)\,\omega\,\{cxl + cym - (ax + by)\,n\} = d\Psi_2/dn,$$

the outside medium being air. If the dielectric body is a sphere rotating in a uniform field $(0, 0, c)$ parallel to the axis, this gives by the usual harmonic analysis $\Psi_1 = \frac{1}{3}(1 - \mathrm{K}^{-1})\,\omega cr^2 + \mathrm{A}r\cos\theta + \mathrm{A}'$ and $\Psi_2 = \mathrm{B}r^{-2}\cos\theta + \mathrm{B}'r^{-1}$, where, r_1 being the radius, $\mathrm{A} = \mathrm{B}/r_1^3 = -3\mathrm{K}/(2\mathrm{K}+1)\,r_1 \,.\, \mathrm{A}' = -\frac{3}{2}/(\mathrm{K}+2)\,r_1^2 \,.\, \mathrm{B}' = (\mathrm{K}-1)/(\mathrm{K}+2) \,.\, \omega cr_1$; thus determining the electric potential Ψ_2 in the space surrounding the rotating sphere.

15. More generally, let us consider steady distributions of electric charges on a system of conductors and dielectric bodies in motion through the æther. That there may be a steady state, without conduction currents, it is necessary that the configuration of the matter shall be permanent, and that its motion shall be the same at all times relative to this configuration and to the æther, and also to the extraneous magnetic field if there is one: this confines it to uniform spiral motion on a definite axis fixed in the æther. Referring to axes fixed in the material system, the vector potential has in the steady motion no time-variation: hence

$$(\mathrm{P}, \mathrm{Q}, \mathrm{R}) = -(d/dx, d/dy, d/dz)\mathrm{V},\ (\mathrm{P}', \mathrm{Q}', \mathrm{R}') = (\mathrm{P} - qc + rb, \mathrm{Q} - ra + pc, \mathrm{R} - pb + qa).$$

The magnetic induction through any circuit moving with the matter being constant, (P, Q, R) is derived (§12) from an electric potential function V. Inside a conductor the electric force must vanish, otherwise electric separation would be going on, therefore V must there be constant.

When the surrounding dielectric is free space, the total current in it, referred to these axes moving with the matter, is $-(p d/dx + q d/dy + r d/dz)\,(f, g, h)$. When the velocity (p, q, r) of the matter is uniform, it then follows from AMPÈRE'S circuital relation that $(a, b, c) = 4\pi\,(qh - rg,\ rf - ph,\ pg - qf)$. Hence (f, g, h), given by $4\pi c^2 f = \mathrm{P} - qc + rb$, is expressed in terms of (P, Q, R) by equations of type $(c^2 - p^2 - q^2 - r^2)\,f = \mathrm{P}/4\pi - p/4\pi c^2\,(p\mathrm{P} + q\mathrm{Q} + r\mathrm{R})$. The circuital quality of (f, g, h) thus gives the characteristic equation of the single independent variable

V of the problem, in the form $\nabla^2 \mathrm{V} = \mathrm{c}^{-2}(pd/dx + qd/dy + rd/dz)^2 \mathrm{V}$, the boundary condition being that V is constant over each conductor.

Thus in the case of a system of conductors moving steadily through space with uniform velocity v in the direction of the axis of x, ϵ denoting $(1 - v^2/\mathrm{c}^2)^{-1}$ we have $(f, g, h) = (4\pi\mathrm{c}^2)^{-1}(\mathrm{P}, \epsilon\mathrm{Q}, \epsilon\mathrm{R})$, and therefore $(d^2/dx^2 + \epsilon\, d^2/dy^2 + \epsilon\, d^2dz^2)\, \mathrm{V} = 0$. The distribution of electric force is therefore precisely the same as if the system were at rest, and the isotropic dielectric constant unity of the surrounding space changed into an æolotropic one $(1, \epsilon, \epsilon)$, *cf.* Part I. §115 ; and so would the surface density of true charge, which is the superficial discontinuity of total displacement, be the same, were it not that there is æthereal displacement *inside* the conductors which must be subtracted. The internal displacement current thence arising is $-(4\pi\mathrm{c}^2)^{-1}\, v d/dx\, (0, -vc, vb)$; hence (a, b, c) is of the form $\{d/dx, (1 + v^2/\mathrm{c}^2)^{-1} d/dy, (1 + v^2/\mathrm{c}^2)^{-1} d/dz\}\, \phi$, by AMPÈRE'S circuital relation : the circuitality of (a, b, c) then leads to a characteristic equation for ϕ, which must be solved so as to give at the surface of the conductor a value for the normal component of (a, b, c) continuous with the already known outside value, and the internal displacement is thereby determined. There is no bodily electrification inside the conductors, since this displacement is circuital.

We can restore the above characteristic equation of V, the potential of the electric force, to an isotropic form by a geometrical strain of the system and the surrounding space, represented by $(x', y', z') = (\epsilon^{\frac{1}{2}}x, y, z)$: the actual distribution of potential around the original system in motion corresponds then to that isotropic distribution of potential round the new system at rest which has the same values over the conductors. The æthereal displacements through related elements of area $\delta\mathrm{S}$ and $\delta\mathrm{S}'$, of direction cosines (l, m, n) and (l', m', n') in the two spaces, multiplied by $4\pi\mathrm{c}^2$, will be

$$-(ld/dx + \epsilon\, md/dy + \epsilon\, nd/dz)\, \mathrm{V}\delta\mathrm{S} \quad \text{and} \quad -(l'd/dx' + m'd/dy' + n'd/dz')\, \mathrm{V}'\delta\mathrm{S}' ;$$

of these the second is always $\epsilon^{-\frac{1}{2}}$ times the first ; thus the elements of surface for which the total displacement is null correspond in the two systems, and therefore the lines and tubes of total displacement also correspond, the flux of displacement in these tubes being $\epsilon^{-\frac{1}{2}}$ times greater in the second system than in the first. But on account of the æthereal displacement in the interior, the outside tubes do not mark out the distribution of the charge on each conductor. If then a system of charged conductors has a velocity of uniform translation v through the æther : and an auxiliary system at rest is imagined consisting of the original system and its space each uniformly expanded in the ratio $\epsilon^{\frac{1}{2}}$ or $(1 - v^2/\mathrm{c}^2)^{\frac{1}{2}}$ in the direction of the motion, and the charges on this new system are $\epsilon^{\frac{1}{2}}$ times those on the actual system : then the fields of æthereal displacement of the two systems agree in the surrounding spaces so as to be the same across corresponding areas, but the distributions of the charges on the conductors do not thus exactly correspond. [These results are obtained on the supposition that the structure of the matter is not affected by its motion. The conductors on which these charges are situated will, however, if the results of the more fundamental analysis of §16 are admitted, change their actual forms to a slight extent depending on $(v/\mathrm{C})^2$ when they are put in motion, and this change will react so that the distribution of charges and displacements will be the simple one there given.]

16. The circumstances of propagation of radiation in a material medium moving with uniform velocity v parallel to the axis of x will form another example. We may here (§13) employ the circuital relations, of types

$$4\pi u = \frac{d\gamma}{dy} - \frac{d\beta}{dz}, \quad -\frac{\delta\alpha}{dt} = \frac{d\mathrm{R}}{dy} - \frac{d\mathrm{Q}}{dz}$$

where

$$u = \frac{df}{dt} + \frac{\delta f'}{dt}, \quad (f', g', h') = \frac{\mathrm{K}-1}{4\pi\mathrm{C}^2}(\mathrm{P}, \mathrm{Q}, \mathrm{R}), \quad (f, g, h) = \frac{1}{4\pi\mathrm{C}^2}(\mathrm{P}, \mathrm{Q} + vc, \mathrm{R} - vb).$$

There readily results, on eliminating the electric force (P, Q, R),

$$4\pi(u, v, w) = \mathrm{curl}\,(\alpha, \beta, \gamma), \quad \mathrm{D}^2/dt^2\,(a, b, c) = 4\pi\mathrm{C}^2\,\mathrm{curl}\,(u, v, w),$$

where

$$\mathrm{D}^2/dt^2 = d^2/dt^2 + (\mathrm{K}-1)(d/dt + v\,d/dx)^2\,;$$

which agrees with the equation obtained in Part I. §124 and Part II. §13, leading to FRESNEL's law of alteration of the velocity of propagation.

Now let us consider the free æther for which K and μ are unity, containing a definite system of electrons which are grouped into the molecules of a material body moving across the æther with uniform velocity v parallel to the axis of x; and let us remove the restriction to steadiness of § 15. We refer the equations of free æther, in which these electrons are situated, to axes moving with the body: the alteration thus produced in the fundamental æthereal equations

$$4\pi d/dt\,.\,(f, g, h) = \mathrm{curl}\,(a, b, c), \quad -d/dt\,.\,(a, b, c) = 4\pi\mathrm{C}^2\,\mathrm{curl}\,(f, g, h)$$

is change of d/dt into $d/dt - v\,d/dx$, leading to the forms

$$4\pi d/dt\,.\,(f, g, h) = \mathrm{curl}\,(a', b', c'), \quad -d/dt\,.\,(a, b, c) = 4\pi\mathrm{C}^2\,\mathrm{curl}\,(f', g', h'),$$

where

$$(a', b', c') = (a, b + 4\pi vh, c - 4\pi vg), \quad (f', g', h'), = (f, g - v\,c/4\pi\mathrm{C}^2, h + v\,b/4\pi\mathrm{C}^2)\,;$$

from which eliminating the unaccented letters, *neglecting* $(v/\mathrm{C})^3$, and writing as before ϵ for $(1 - v^2/\mathrm{C}^2)^{-1}$, we derive the system

$$\begin{aligned}
4\pi\frac{df'}{dt} &= \frac{dc'}{dy} - \frac{db'}{dz} & -(4\pi\mathrm{C}^2)^{-1}\frac{da'}{dt} &= \frac{dh'}{dy} - \frac{dg'}{dz} \\
4\pi\epsilon\frac{dg'}{dt} &= \frac{da'}{dz} - \left(\frac{d}{dx} + \frac{v}{\mathrm{C}^2}\frac{d}{dt}\right)c' & -(4\pi\mathrm{C}^2)^{-1}\epsilon\frac{db'}{dt} &= \frac{df'}{dz} - \left(\frac{d}{dx} + \frac{v}{\mathrm{C}^2}\frac{d}{dt}\right)h' \\
4\pi\epsilon\frac{dh'}{dt} &= \left(\frac{d}{dx} + \frac{v}{\mathrm{C}^2}\frac{d}{dt}\right)b' - \frac{da'}{dy} & -(4\pi\mathrm{C}^2)^{-1}\epsilon\frac{dc'}{dt} &= \left(\frac{d}{dx} + \frac{v}{\mathrm{C}^2}\frac{d}{dt}\right)g' - \frac{df'}{dy}.
\end{aligned}$$

Now change the time variable from t to t', equal to $t - vx/\mathrm{C}^2$, so that $d/dx + v/\mathrm{C}^2 . d/dt$ becomes d/dx, and d/dt becomes d/dt', and these equations assume the form of an electric scheme for a crystalline medium at rest. Finally write x_1 for $x\epsilon^{\frac{1}{2}}$, a_1 for $a'\epsilon^{-\frac{1}{2}}$, f_1 for $f'\epsilon^{-\frac{1}{2}}$, dt_1 for $dt'\epsilon^{-\frac{1}{2}}$, keeping the other variables unchanged, and the system comes back to its original isotropic form for free æther. Thus the final variables (f_1, g_1, h_1) and (a_1, b_1, c_1) will represent the æthereal field for a correlative system of electrons forming the molecules of another material system at rest in the æther, of the form of the original one pulled out uniformly in the ratio $\epsilon^{\frac{1}{2}}$ along its direction of movement; the electric displacements through corresponding areas in the two systems are not equal, but their molecules are composed of equal electrons and are situated at corresponding points, and the individual electrons describe corresponding parts of their orbits in times shorter for the latter system in the ratio $\epsilon^{-\frac{1}{2}}$ or $(1 - \frac{1}{2}v^2/\mathrm{C}^2)$, while those less advanced in the direction of v are also relatively very slightly further on in their orbits on account of the difference of time-reckoning. Thus we have here two correlative systems each governed by the circuital relations of the free æther: (i) a system in which the electric and magnetic displacements are (f, g, h) and (a, b, c), moving steadily with uniform velocity v parallel to the axis of x, (ii) the same system expanded in the direction of x in the ratio $\epsilon^{\frac{1}{2}}$ and at rest, the displacements at the corresponding points being $(\epsilon^{-\frac{1}{2}}f,\ g - vc/4\pi\mathrm{C}^2,\ h + vb/4\pi\mathrm{C}^2)$ and $(\epsilon^{-\frac{1}{2}}a,\ b - 4\pi vh,\ c + 4\pi vg)$, and the molecules being situated in the corresponding positions with due regard to the varying time-origin. Inasmuch as the circuital relations form a differential scheme of the first order which determines step by step the subsequent stages of a system when its initial state is given, it follows that if these two æthereal systems are set free at any instant in corresponding states, they will be in corresponding states at each subsequent instant, their electrons or singularities being at corresponding points. If then the latter collocation represent a fixed solid body, the former will represent the same body in uniform motion; one consequence of the motion thus being that the body is shrunk in the direction of its velocity v in the ratio $\epsilon^{-\frac{1}{2}}$ or $1 - \frac{1}{2}v^2/\mathrm{C}^2$. It may be observed that there is here no question of verifying that the mechanical forces acting on the single electrons in the two cases are such as to maintain this correspondence; for in the present complete survey of the individual atoms there is no such entity as mechanical force, any more than there is on a free vortex ring in fluid; the notion of mechanical forces enters at a subsequent stage when we are treating of molecular aggregates considered as continuous bodies, and are examining the relations between the different groups into which our senses analyze their interactions (§ 48).

If this argument is valid, it will confirm the hypothesis of FITZGERALD and LORENTZ, to which they were led as the ultimate resource for the explanation of the negative result of MICHELSON's optical experiments; and conversely it will involve evidence that the constitution of a molecule is wholly electric, as here represented.

The reasoning given in Part II., § 13, was insufficient, because the correlation between the two systems was not there pushed to their individual molecules.

Consideration of a possible Limitation of the Rotational Scheme.

17. In the preceding sections the equations of the æthereal field have been expressed (as they were in Part II.) without reference to the dynamical hypothesis of a rotational æther which suggested their present form. Reverting now to that hypothesis, let us examine whether a limitation of the kind that was unavoidable in the material model of § 3, may not also be involved in the general scheme of a rotational æther. In the first place, it is natural to take the elastic rotation in the medium as very small, so that its translatory velocity which is connected therewith is also very small, though the velocities of the strain-centres which flit across it, and represent the matter, may have any values; this is in agreement with the conclusion derived from optical experiments that the æther is practically stagnant. But there is one conceivable class of cases in which the changes of position of the elements of the medium go on accumulating, that namely of a steady magnetic field kept up say by a current of electrons constrained to flow permanently round a circuit. On account of the smallness of the velocity of the æther, the corrections to the dynamical equations which arise from the velocity of convection of the elastic strain may always be left out of account, being utterly insignificant in ordinary electrodynamics, and actually beyond the limits of experiment in optics: yet in a magnetic field continuing steady *for an unlimited time* the elements of volume of the æther will ultimately have wandered far from their original positions, and a difficulty presents itself.* To cover such a case, the definition of the elastic rotation of the medium must be made more precise. For the motion of a perfect fluid, which is differentially irrotational at each instant, will yet result after a time in finite rotations of its elements of volume; for example it is known that if a rigid ellipsoidal shell be filled with perfect fluid, and be set rotating about a fixed axis,

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* I am indebted to Lord RAYLEIGH for drawing my attention to this point, as one requiring further consideration.

[A steady magnetic field involves a cyclical motion of the æther; thus in a very great time even a very small velocity will produce large changes of position. It is true that any motion of electrons whatever will produce change of strain, and therefore movement in the æther, but that movement will be very slight, and will not be cumulative except in the one case of permanent cyclical motion which represents absolutely permanent magnets. If there were no other way out of the difficulty described in the text, it might be turned by simply asserting that absolutely permanent magnets do not exist.

The nature of the constraints which may be necessary to prevent the nucleus of an electron from ever becoming sub-divided is a different question, and wholly outside the scope of the present theory, which simply takes these nuclei to exist as it finds them without inquiring in detail into their structure.]

then after a certain interval of time the parts of the fluid will have returned to the original configuration with respect to the shell, so that the fluid will have been rotated bodily in space just like a solid, although its motion at each instant has been differentially irrotational. In fact when the change of position of the element of volume is finite we can no longer analyze it definitely into rotations and pure strains, in such wise that the order of their application shall be indifferent; thus we obtain no longer in that way an analytical specification of the æthereal elastic rotation, and a more precise formulation must be made. The rough material model of § 3 indicates the necessary modification: in that model a differential pure strain of the element of volume does not tend to rotate the sub-element which is elastically effective; thus the efficient elastic rotation is the vector sum of the series of differential rotations which the element of the æther has experienced in its previous history. This is therefore the more precise definition of the total rotation, proportional to (f, g, h), from which the electrostatic forcive is derived as in Part II., § 18: it makes the rotation equal to the curl of the linear displacement when these quantities are both small so that their squares and products can be neglected, but not after the long-continued cumulative effect of a permanent magnetic field has come in. In that case, however, the small irrotational velocity, say for the moment (p, q, r), which constitutes the magnetic field, will contribute to (f, g, h) only by shifting by convection the element of the medium along with its rotation, while the rotation so transferred will be continually re-adjusting itself by elastic action into the new equilibrium configuration: the relation between elastic rotation and magnetic force will then be of the type $df/dt = d\gamma/dy - d\beta/dz - (pd/dx + qd/dy + rd/dz)f$, where (p, q, r) is equal to (α, β, γ) multiplied by a very small scalar factor. Unless the velocity (p, q, r) is uniform, $d/dt\,(df/dx + dg/dy + dh/dz)$ will not be exactly null; so that the movement of the æther by the steady magnetic field will lead to a development of electric charge, extremely slow and gradual, throughout the volume concerned. On the other hand the combination of permanent electric and magnetic fields which is the origin of such a creation of electricity must be confined to a limited region, beyond which the æther is in equilibrium; therefore the electrification thus developed consists of compensating amounts of positive and negative signs. These diffuse charges, of the second order of small quantities, will subsequently by their mutual attractions drift together again and neutralize each other, by moving as strain-forms across the æther without sensibly interfering with the motion of the medium itself (§ 6). Thus a steady magnetic field of unlimited duration would not theoretically get interlocked with a concomitant electrostatic field, but would relieve itself by very slowly developing a very minute diffuse electrification which will simultaneously gradually fade away by its own natural actions, so that no sensible effect would ever be accumulated. The rotational æther scheme therefore would not break down in this limiting case, the consequence of long-continued cumulation being obviated by a process which is at each instant so insignificant as to be far below the reach of experience: electrons may, it is true, also conceivably obliterate each other in the same way as these diffuse electrifications, but that is a contingency of negligible probability with which we are familiar in all kinds of molecular theory.

Relations of Inductive Capacity and Optical Refraction to Density.

18. Let the medium be free æther containing n similar molecules per unit volume; and suppose each molecule to be polarized to moment μ by the field of electric force. This field is made up of the extraneous exciting field and that of the polarized molecules themselves; the latter again consists of a part arising from the polarized medium as a whole and a part involving only the immediate surroundings of the point considered. If h denote this local part, and H the remainder of the total electric field, we have relations of the types $\mu = k(\mathrm{H}+h)$, $i' = n\mu$, $h = \lambda . n\mu$, i' denoting the intensity of the polarization, k a constant independent of the density of the material medium, and λ a parameter which, as will appear, must be nearly independent of the density. These relations lead to $i' = kn(\mathrm{H} + \lambda i')$, that is, since by the definition of the inductive capacity K, $i' = (\mathrm{K} - 1)/4\pi . \mathrm{H}$ with electrostatic units, they lead to $3/4\pi . kn = (\mathrm{K} - 1)/(\mathrm{K} - 1 + 4\pi/\lambda)$; so that, ρ denoting density, $(\mathrm{K} - 1)/(\mathrm{K} - 1 + 4\pi/\lambda)\,\rho$ is constant for the same material medium. For fluid media at any rate, it will appear that λ must be very nearly equal to $\frac{4}{3}\pi$, so that for them LORENTZ's expression $(\mathrm{K} - 1)/(\mathrm{K} + 2)\,\rho$ should be approximately constant.

When the dielectric is a compound one consisting of n molecules of one kind and n' of another per unit volume, we have $i' = n\mu + n'\mu'$, $h = \lambda i'$; and $\mu = k(\mathrm{H} + h)$, $\mu' = k'(\mathrm{H} + h)$, so that $\mu/k = \mu'/k' = i'/(kn + k'n')$. Thus $i' = (kn + k'n')(\mathrm{H} + \lambda i')$, so that, with the above value $\frac{4}{3}\pi$ for λ, $3/4\pi . (kn + k'n') = (\mathrm{K} - 1)/(\mathrm{K} + 2)$. This formula gives an additive character to the refraction equivalent for a mixture; and also for a compound body, provided in the latter case the moment μ belongs to the individual atom, and is not sensibly affected by the molecular grouping of the atoms.

This investigation is of course not absolutely exact; but it is the first approximation in a statistical theory, and the question presents itself how far it is a sufficient approximation. On examination, it will appear that the coefficients k and k' are rightly taken to be numerical quantities independent of n and n', provided the distance between the effective poles of an atom or molecule is not a considerable fraction of the mean distance between adjacent molecules. The constancy of the value of λ, when the component densities are altered, appears from considerations of dimensions. For the force due to a polarized molecule varies as $\mu \times (\text{distance})^{-3}$: thus, as on change of density $(\text{distance})^{-3}$ varies directly as density, the character of the arrangement of the molecules being supposed unaffected, the force due to the molecules surrounding the point is proportional to $\mu \times$ density, that is, it is equal to $\lambda n\mu$. For the case of a mixture λ is the same for both constituents; a result which may or may not hold good for a solution or a chemical compound.

19. The value of λ, namely $\frac{4}{3}\pi$, which has here been assumed, is not merely dictated by the form desired for the final result. That value has in fact already been specified as the first approximation in quite another connexion.* As this is the crucial point of the theory, it may be allowable to present the argument in detail. The total electric force acting on a single molecule is derived from the aggregate potential $V = \Sigma (\mu_x d/dx + \mu_y d/dy + \mu_z d/dz) r^{-1}$, where μ_x, μ_y, μ_z are the components of the moment μ of a polarized molecule. This potential, when the point considered is inside the polarized medium, involves the actual distribution of the surrounding molecules; and thus the force derived from it changes rapidly at any instant of time, in the interstices between the molecules. But when the point considered is outside the polarized medium, or inside a cavity formed in it whose dimensions are considerable compared with molecular distances, the summation in the expression for V may be replaced by continuous integration; so that, (f', g', h') being the intensity of polarization in the molecules of the dielectric,

$$V = \int (f' \, d/dx + g' \, d/dy + h' \, d/dz) \, r^{-1} \, d\tau ;$$

and the force thence derived is perfectly regular and continuous. This expression may be integrated by parts, since, the origin being outside the region of the integral, no infinities of the function to be integrated occur in that region. Thus

$$V = \int (lf' + mg' + nh') \, r^{-1} \, dS - \int (df'/dx + dg'/dy + dh'/dz) \, r^{-1} \, d\tau ;$$

that is, the potential at points in free æther is due to POISSON's ideal volume density ρ, equal to $-(df'/dx + dg'/dy + dh'/dz)$, and surface density σ, equal to $lf' + mg' + nh'$. When the point considered is in an interior cavity, this surface density is extended over the surface of the cavity as well as over the outer boundary. Now when it is borne in mind that, at any rate in a fluid, the polar molecules are in rapid movement, and not in fixed positions which would imply a sort of crystalline structure, it follows that the electric force on a molecule in the interior of the material medium, with which we are concerned, is an average force involving the average distribution of these polar molecules, and is therefore properly due to an ideal continuous density like POISSON's, even as regards elements of volume which are very close up to the point considered. To compute the average force which causes the polarization of a given molecule we have thus to consider that molecule as situated in the centre of a spherical cavity whose radius is of the order of molecular distances; and we have to take account of the effect of a POISSON distribution on the surface of this cavity, or more precisely of the result of an averaged continuous local polarization, surrounding the molecule, whose intensity increases from nothing at a certain distance from the centre up to the full amount i'

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* "On the Theory of Electrodynamics," 'Proc. Roy. Soc.,' 52 (1892), p. 64.

at the limit of the molecular range, this intensity being uniform in direction and a function of the distance only. The force due to this is $\frac{4}{3}\pi i'$ along the direction of the polarization i'; which is therefore the local part to be added on to the electric force as ordinarily defined—namely, to that arising from the density ρ throughout the medium and the density σ on its external surface, and so everywhere derivable from a potential by the theory of gravitational mass-distributions. The value of the coefficient λ of the above analysis should thus be $\frac{4}{3}\pi$ for a fluid;[*] but it may deviate from this value somewhat in the case of a solid, especially of course if it be crystalline.

20. The mathematical principles, on which the above formula for the relation between inductive capacity and density is based, were first given by POISSON for the corresponding problem in magnetic polarization. The explicit application to electric polarization, on the lines of FARADAY'S ideas, was made by Lord KELVIN and MOSSOTTI. The investigation of the formula which has been implicitly given by MAXWELL ("Treatise," § 313), expressed in terms of the cognate problem of conduction, is however valid only for the case in which the coefficient of polarization of the medium is small compared with unity, that is, only for gaseous media. The same formula, viewed as a relation between refractive index and density for transparent media, was obtained by LORENTZ[†] and was shown by him to be experimentally valid in an approximate way over the wide range of density including the liquid and gaseous states; though for the small changes of density induced in a liquid by alterations of pressure and temperature the effect of the change in the internal energy and mutual configuration of the molecules may considerably mask the direct effect of the slight change of density.[‡] For gases, however, in which the molecules are more isolated and the changes of density greater, the refraction is found to be in accordance with the formula. The investigation of LORENTZ[§] was probably the first effective attempt to introduce the molecular constitution of the medium into the electric theory of light, and so arrive at laws of refraction and dispersion. The *form* of the refraction constant was really settled by statical considerations akin to those here given; but the theory of electric propagation employed by him at that

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[*] The fact that the values of the refractive index for liquids are slightly in excess of what LORENTZ'S formula would give by computation from the values for their vapours, may be an indication that this averaged field of molecular action is slightly elongated instead of spherical.

[†] H. A. LORENTZ, "Ueber die Beziehung zwischen der Fortpflanzungsgeschwindigkeit des Lichtes und der Körperdichte." 'Wied. Ann.,' 9, 1879, p. 641.

[‡] For these small changes, the LORENTZ refraction function $(m^2 - 1)/(m^2 + 2)$ is approximately proportional to that of GLADSTONE and DALE, their ratio $(m + 1)/(m^2 + 2)$ being nearly constant; but it does not appear why the latter function happens to be usually more nearly proportional to the density than the former. The results of RÖNTGEN and ZEHNDER, 'Wied Ann.,' 44, 1891, on the effects of pressure on various fluids, make the two formulæ in default in opposite directions by about equal amounts.

[§] 'Verhandl. der Akad. Amsterdam,' 18; abstract in 'Wied Ann.,' 9, 1872, pp. 641–665.

time was the one developed in VON **HELMHOLTZ**'s early memoirs on electrodynamics, and it would appear that discrepancies come in through treating the æther as polarized like a material dielectric; at any rate his final result (*loc. cit.* p. 654) seems to give the refractive index a value greater than unity for free æther, and one only infinitesimally different for a ponderable medium. A mathematical investigation has been given by Lord **RAYLEIGH**,* in which the range of density over which these statical computations are valid is tested by finding for certain cases the complete expressions in a statical theory, of which they form the first rough approximations. The result is rather unfavourable to **LORENTZ**'s formula, so much so as perhaps to excite surprise at its close agreement with the facts when the range of density is so great as that between the liquid and gaseous states of the same substance. There is thus room for the statistical method under which the subject has here been approached,† in that it explains the wide range through which the formula proves to be valid as a first approximation; while it at the same time recognizes that when the change of density is itself small, but is accompanied by other kinds of physical change, the influence of the latter on the polar molecule may be sufficiently important to prevent its exact verification.‡ On the specific influence of temperature, *cf.* § 72 *infra.*

21. In thus basing a theory of refraction equivalents on the value of the inductive capacity, it has been tacitly assumed that the dispersion of the medium is small; hence the results apply certainly only in the cases in which there is approximate agreement between the inductive capacity and the square of the refractive index.§ When dispersion in *absolutely* non-conducting media is taken into account, as in the previous memoir, § 11, and *infra*, § 23, the formula however still holds, the constant κ, equal to f'/P of § 24, now involving the period of the light.

The fact that for gases, and a large class of denser bodies as well, the inductive capacity is approximately equal to the square of the refractive index, shows that in them the polarization of the molecules can completely follow the rapid alternations of electric force which belong to the light waves. Thus we can conclude that when the

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* RAYLEIGH, "On the influence of obstacles arranged in rectangular order on the properties of a medium," 'Phil. Mag.,' 34, 1892 (2), p. 481.

† Since this was written, I have found that the analytical method here employed is essentially the same as that of CLAUSIUS ('Mechanische Wärmetheorie,' 2, 1879); the fundamental importance of the ideas involved, and the discussion here given of the value of λ, in the case of fluid media, may perhaps justify the retention of the above independent statement.

‡ A theory precisely similar to the above of course applies to determinations of molecular magnetization in solutions of iron or other salts; strictly it is not the coeffcient of magnetization κ, but $\kappa/(1 + \frac{4}{3}\pi\kappa)$ that is proportional to the density of the magnetic molecules. The values of κ are however usually so small that this constant is practically equal to κ.

§ This accords with the conclusion drawn by LINDE from an experimental examination of the subject, 'Wied. Ann.,' 56, 1895, pp. 546–70 (see p. 566). [PHILIP, 'Zeitsch. Phys. Chem.,' 1897, finds that the CLAUSIUS formula is quite inapplicable to mixtures of substances with abnormally high values of K.]

polarization of a molecule is upset by an encounter with another molecule, it is instantly restored to its normal value, as soon as the violence of the encounter is over; so that, the relative times spent by the molecule in encounters being small in every case, they hardly affect the inductive capacity of the medium; or in other terms, the density by itself hardly affects the molecular refraction equivalent (except in so far as the restoration of the steady state may involve absorption, § 28 *infra*), and the constancy of the coefficient k is further justified.*

22. The molecular theory leads to the conclusion that the electric æolotropy of crystals in which the dielectric constant differs much from unity, may be in part due to the distribution of the molecules in space and in part to the orientation of the individual molecules; and that therefore the same applies to double refraction. The intrinsic polarity which is revealed by pyroelectricity and piezoelectricity also shows that orientation is a real cause. But magnetic æolotropy must practically be wholly due to orientation of the molecules, as the smallness of the susceptibility makes the effect of arrangement inappreciable. The double refraction induced in dielectrics in a strong electric field is possibly mainly due to molecular orientation, as also that arising from mechanical strain.

The difference of absorption in different directions in a crystal like tourmaline must be of an order of numerical magnitude not higher than the difference of the refractions: an easy computation shows that it is really of a considerably lower order. This crystalline absorption can only be due to molecular orientation: it is of course excessively smaller than the absorption in metals, which is comparable with the whole refractive index; it would not therefore sensibly affect the laws of reflexion.

General Theory of Optical Dispersion.

23. A formula for optical dispersion was obtained in § 11 of the second part of this memoir, on the simple hypothesis that the electric polarization of the molecules vibrated as a whole in unison with the electric field of the radiation. The kinetic molecule of § 11 *supra*, with its steady momenta, will however usually have various free periods, and as many absorption bands; to take account of them, and also for other reasons which will appear, it is desirable to have a more complete dynamical theory.

The problem of dispersion, in its general form, is thus that of the transmission

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* The analysis of this section does not agree with a theoretical investigation of the inductive capacities of mixtures of non-conducting liquids which do not exhibit change of volume in mixing, given by SILBERSTEIN ('Wied. Ann.,' 1895); his result is that K, or what comes to the same under these conditions, $K-1$, is an additive physical constant, whereas the formula of CLAUSIUS and LORENTZ makes $(K-1)/(K+2)$ additive. The determinations made by SILBERSTEIN for mixtures of benzol and phenylethylacetate give results for the LORENTZ constant which are always in excess of the theoretical value, by amounts ranging up to 8 per cent.; the discrepancies for his own constant $K-1$ are rather smaller, and are in both directions.

of radiation across a medium permeated by molecules, each consisting of a system of electrons in steady orbital motion, and each capable of free oscillations about the steady state of motion with definite free periods analogous to those of the planetary inequalities of the Solar System; and its analysis will in fact resemble LAPLACE'S general investigation of the latter problem. If $\theta_1, \theta_2, \ldots$ represent small deviations from the state of steady motion of a molecule, so that the coordinates of the system are $f_1(t) + \theta_1, f_2(t) + \theta_2, \ldots$, the kinetic and potential energy of the molecule when expanded in powers of these small quantities will assume the forms

$$\begin{aligned} \mathrm{T} = \text{const.} &+ [\theta_1, \theta_2, \ldots]_1 + [\dot\theta_1, \dot\theta_2, \ldots]_1 + [\theta_1, \theta_2, \ldots]_2 + [\dot\theta_1, \dot\theta_2, \ldots]_2 \\ &+ [\{\theta_1, \theta_2, \ldots\} \{\dot\theta_1, \dot\theta_2, \ldots\}] \\ \mathrm{W} = \text{const.} &+ [\theta_1, \theta_2, \ldots]_1 + [\theta_1, \theta_2, \ldots]_2, \end{aligned}$$

where the terms in T and W denote functions of the various degrees of these velocities and displacements, the last term in T being a lineo-linear function of them jointly. From these expressions the free vibrations are determined by the LAGRANGIAN method. As the undisturbed motion is steady, the type of a free vibration must be the same at whatever time it is excited, therefore the coefficients in T and W are all independent of the time; indeed if they were not constant the system could have no free periodic vibrations at all. The equations of the steady motion show that there can be no terms in T − W of the first degree in the displacements, when the coordinates are properly chosen.* At the present stage we may conveniently by transformation of coordinates express the LAGRANGIAN function, on which the motion in the molecule depends, in the form

$$\begin{aligned} \mathrm{T} - \mathrm{W} = [\dot\theta_1, \dot\theta_2, \ldots. \dot\theta_n]_1 + \tfrac{1}{2}\{\mathrm{A}_1\dot\theta_1^2 + \mathrm{A}_2\dot\theta_2^2 + \ldots. + \mathrm{A}_n\dot\theta_n^2\} \\ - \tfrac{1}{2}\{a_1\theta_1^2 + a_2\theta_2^2 + \ldots. + a_n\theta_n^2\} + b_{11}\theta_1\dot\theta_1 + \ldots. + b_{12}\theta_1\dot\theta_2 + b_{21}\theta_2\dot\theta_1 + \ldots., \end{aligned}$$

from which, by a property which is an immediate corollary of the Action principle, we may subtract any perfect differential coefficient with respect to time, for example here

$$[\dot\theta_1, \dot\theta_2, \ldots. \dot\theta_n]_1 + d/dt\,\{\tfrac{1}{2}b_{11}\theta_1^2 + \ldots. + \tfrac{1}{2}(b_{12} + b_{21})\,\theta_1\theta_2 + \ldots.\},$$

without affecting the course of the motion, leaving thus an *effective Lagrangian function*

$$\begin{aligned} \mathrm{L} = \tfrac{1}{2}\{\mathrm{A}_1\dot\theta_1^2 + \mathrm{A}_2\dot\theta_2^2 + \ldots. + \mathrm{A}_n\dot\theta_n^2\} - \tfrac{1}{2}\{a_1\theta_1^2 + a_2\theta_2^2 + \ldots. + a_n\theta_n^2\} \\ + \tfrac{1}{2}\{e_{12}(\dot\theta_1\theta_2 - \dot\theta_2\theta_1) + \ldots.\}. \end{aligned}$$

24. We have now to determine the vibrations forced on this molecule by the uniform alternating field of electric force, say P parallel to the x axis, belonging to the radiation which is traversing the medium. Bearing in mind that the wave length covers about 10^3 molecules, it appears that if f' denote the total intensity

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* This analysis, so far, is as given by ROUTH, "Advanced Rigid Dynamics," § 111.

per unit volume of polarization of the molecules, the electric force acting on a single molecule will, as in § 19 but now using electrodynamic units, be $P_1 = P + \lambda c^2 f'$; this force will maintain vibratory motion in the polar molecule, but will not cause any oscillation of its centre of mass. The interaction of the electric field with the internal coordinates of the molecule will thus introduce an extraneous potential energy function of the form

$$W' = F(t) - (c_1\theta_1 + c_2\theta_2 + \ldots\ldots + c_n\theta_n)\,P_1,$$

higher powers of the small internal coordinates $\theta_1, \theta_2, \ldots\ldots \theta_n$ being, as usual in problems of vibration, omitted; and here again the coefficients $c_1, c_2, \ldots\ldots c_n$ must be independent of the time. There will also be terms in the kinetic energy involving the interaction of the magnetic intensity of the field with the component velocities of the molecular vibration: now in a train of waves of type exp. $q\,(t - K'^{\frac{1}{2}}c^{-1}z)$, the magnetic induction b, being derived from the electric force P, both in the plane xy of the wave-front, by the relation $-\,db/dt = dP/dz$, is equal to $K'^{\frac{1}{2}}c^{-1}P$: hence this part of the total kinetic energy will be of the form

$$T' = f(t) + (c'_1\dot{\theta}_1 + c'_2\dot{\theta}_2 + \ldots\ldots c'_n\dot{\theta}_n)\,K'^{\frac{1}{2}}c^{-1}P,$$

where $c'_1, c'_2, \ldots\ldots c'_n$ are coefficients independent of the time.

The form of W′ shows that $c_1\theta_1 + c_2\theta_2 + \ldots\ldots + c_n\theta_n$ is equal to the electric polarization f_1 in the molecule on which the electric force P_1 acts. If unit volume of the medium contains n_1 molecules of one kind, n_2 of another and so on, and the polarizations in each molecule are respectively f_1, f_2 and so on, then

$$f' = n_1 f_1 + n_2 f_2 + \ldots\ldots$$

25. To obtain the general equation of propagation in the æther, let $\mathfrak{F}$ denote the electric force, or the torque acting on the æther; and we have, as in Part II. § 11, the kinematic relation $(4\pi)^{-1}$ curl $\mathfrak{B} = d/dt\,(\mathfrak{D} + \mathfrak{D}') + \mathfrak{C}$, and also the dynamical equation $-\,d\mathfrak{B}/dt =$ curl $\mathfrak{F}$ where $4\pi c^2\,\mathfrak{D} = \mathfrak{F}$. It is to be observed that this dynamical equation leaves out the purely local part of the electric force. The propagation of radiation of ordinary wave-length is in fact an action involving the medium in bulk, and not one of molecular type; thus in accordance with the YOUNG-POISSON principle (*infra* § 47) the local part of the electric force, arising from the surrounding molecules, is compensated intermolecularly by an influence on the physical properties of the material medium which thereby become functions of the density and strain, and this part therefore does not enter into the molar electric forcive maintaining the radiation. These equations lead to

$$c^2\nabla^2\mathfrak{D} = d^2/dt^2\,(\mathfrak{D} + \mathfrak{D}') + d\mathfrak{C}/dt.$$

Hence, when the current of conduction $\mathfrak{C}$ is non-existent, $K' = 1 + \mathfrak{D}'/\mathfrak{D}$; whilst here $\mathfrak{D}'$ is f', and $4\pi c^2\mathfrak{D}$, or P, is $P_1 - \lambda c^2 f'$; so that $(K' - 1)/(K' - 1 + 4\pi/\lambda) = \lambda c^2 . f'/P_1$, or taking λ equal to $\frac{4}{3}\pi$, we have $(K' - 1)/(K' + 2) = \frac{4}{3}\pi c^2 . f'/P_1$.

26. The value of f'/P_1 is to be obtained from the equations of forced vibration of

the molecules. By the LAGRANGIAN method, these equations expressed for a molecule of the first kind and for radiation of the above type e^{qt}, form a system, skew symmetric in so far as $e_{21} = -e_{12}$, of type

$$(A_1q^2 + a_1)\,\theta_1 + e_{12}q\theta_2 + \dots\dots + e_{1n}q\theta_n - c_1P_1 + c'_1K'^{\frac{1}{2}}C^{-1}qP = 0,$$

wherein

$$c_1\theta_1 + c_2\theta_2 + \dots\dots + c_n\theta_n - f_1 = 0.$$

They give the relation

$$f_1 = [\{A_1q^2 + a_1, e_{12}q, \dots\dots, e_{1n}q, -c_1\}\,P_1 + K'^{\frac{1}{2}}C^{-1}q\,\{A_1q^2 + a_1, e_{12}q, \dots\dots, e_{1n}q, c'_1\}\,P] \\ \div \{A_1q^2 + a_1, e_{12}q, \dots\dots, e_{1n}q\},$$

in which the denominator represents a skew determinant, and each of the two coefficients in the numerator the same determinant bordered. The denominator involves when expanded only even powers of q, and when equated to zero it gives the periods of the free vibrations in the molecule; as these are all real the roots in q^2 must be all real and negative. The second term in the numerator has C^{-1} as a factor; we may therefore neglect it as has been done in the previous paper; this means that the elasticity of the æther is so high compared with its inertia that the pull exerted by it on the molecule will be important while the interaction of its kinetic energy will be negligible. The remaining determinant in the numerator, when expanded, contains only even powers of q and is of order lower by two than the denominator. Hence writing $-p^2$ for q^2, so that $2\pi/p$ is the period of the radiation, and expanding in partial fractions, we can express the equation in the form

$$4\pi C^2\frac{f_1}{P_1} = \frac{g_1}{p_1^2 - p^2} + \frac{g_2}{p_2^2 - p^2} + \dots\dots + \frac{g_n}{p_n^2 - p^2},$$

in which $g_1, g_2, \dots\dots g_n$ are real quantities, positive or negative.

Now the index of refraction μ or $K'^{\frac{1}{2}}$ of the compound medium is given by the formula $(K' - 1)/(K' + 2) = n_1 \,.\, \frac{4}{3}\pi C^2 f_1/P_1 + n_2 \,.\, \frac{4}{3}\pi C^2 f_2/P_2 + \dots\dots$

The final result is thus

$$\frac{K' - 1}{K' + 2} = \Sigma nm, \text{ where } m = \frac{g_1}{p_1^2 - p^2} + \frac{g_2}{p_2^2 - p^2} + \dots\dots + \frac{g_n}{p_n^2 - p^2};$$

so that it is m and not μ^2 that has an infinity at each free period of the molecule. We here again arrive at LORENTZ's refraction-equivalent, and the theorem that it is an additive physical constant; but with the important addition that it is the law of dispersion of the molecular refraction-equivalent m, equal to $(\mu^2 - 1)/(\mu^2 + 2)\,\rho$, of each constituent of the medium, not that of the refractive index of the aggregate, which admits of simple theoretical expression. In physical investigations concerning laws of dispersion, it is thus essential to deal with simple substances; the dispersion in the molecular refraction-constant of a mixture, and no doubt also to some extent of a solution or chemical compound, is made up, according to this formula, of the aggregate of those of its constituents.*

27. Let us consider briefly the case of a perfectly transparent substance whose dispersion is dominated by a single free period, say $2\pi/p_1$: the equation is

$$\frac{\mu^2-1}{\mu^2+2}=\frac{ng_1}{p_1^2-p^2},\quad \text{that is,}\quad \mu^2=1+3\,\frac{ng_1}{p_1^2-p^2}\Big/\left(1-\frac{ng_1}{p_1^2-p^2}\right).$$

It will be convenient to form a graph of the formula for μ^2; when p is small, μ^2 has a positive value, which should be the statical dielectric constant of the material; as p increases, μ^2 increases until it becomes infinite when $p^2=p_1^2-ng_1$; it then becomes negative, but again attains a positive value after $p^2=p_1^2+2ng_1$ which corresponds to value zero. Thus there is a band of absorption, which is absolutely complete for some distances on both sides of the bright spectral line corresponding to the substance in the gaseous state, but which extends about twice as far on the upper side of that line as it does on the lower when ng_1 is positive, as will be the case when μ exceeds unity and the dispersion is in the normal direction. When, as in all ordinary media, the dispersion of the visible light is small, being for example of the order of one per cent. for glass, p_1 must be great compared with p, and the range of this single dominant ultra-violet band of absolutely complete absorption would be measured by an interval $(\delta p/p)$ equal to $\frac{1}{2}(\mu^2-1)/(\mu^2+2)$ below the free period, and one equal to $(\mu^2-1)/(\mu^2+2)$ above it, where μ is the index for luminous rays.

28. For a substance such as a gas, with numerous narrow bands of absorption, in the immediate neighbourhood of any one of them the value of μ^2 depends on that one alone; the breadth of the band of complete absorption thus corresponds to a total interval ($\delta p/p$ or $-\delta\lambda/\lambda$) equal to $3ng_1/2p_1^2$, which should thus be proportional to the density of the gas. The distance on each side of the band to which the anomalous dispersion extends, which may possibly be observed as has been done by KUNDT for sodium vapour, ought also to be of the order of magnitude of ng_1/p_1^2. The law of JANSSEN, that the amount of the absorption in a compressed gas is roughly proportional to the square of the density, seems to show that in dense media most of the actual specific absorption is outside these limits of complete blackness, and is conditioned by the molecular encounters deranging the states of steady directed synchronous vibration, say by rotation of the molecule, and so necessitating absorption of fresh energy from the radiation in order to re-establish them. It is to be observed that this process would be a true absorption of radiation which would go to heating the gas, as contrasted with mere refusal of a perfectly transparent gas to transmit radiation in a region in which μ^2 is negative.† The gradual change from an emission

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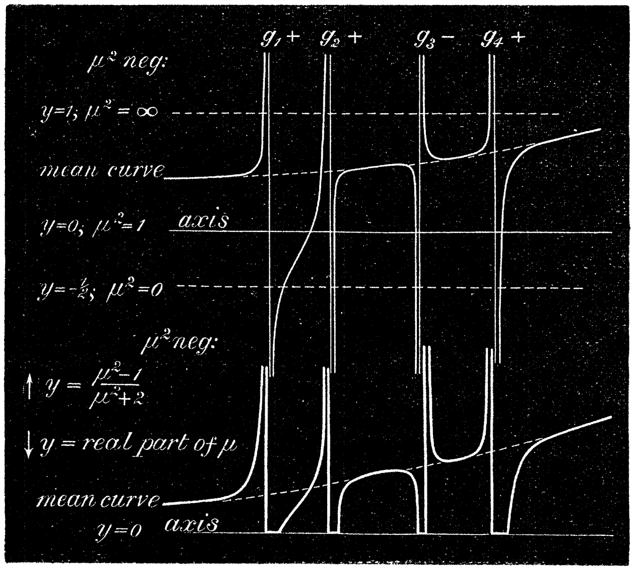
[*] In cases however in which a formula of the CAUCHY type is sufficiently exact, so that $(\mu^2-1)/(\mu^2+2)\,\rho=\mathrm{A}+\mathrm{B}/\lambda^2+\mathrm{C}/\lambda^4+\ldots$, not only is A an additive refraction-equivalent, but there will also be additive dispersion-equivalents B, C, ...

[†] The validity of the general formulæ is not vitiated by this circumstance that the molecules are in

spectrum of definite lines to a continuous spectrum, with increasing density, would thus be due, not to any want of definiteness of the free periods, but to changes in the orientation of the vibrating molecules arising from increased frequency of encounters, the corresponding rather abrupt changes in the radiation received at any point not being analyzable into the regular FOURIER periods.

29. The possible characteristics of the dispersion of an ideal perfectly transparent medium may be very simply represented by a graph of the general formula $(\mu^2 - 1)/(\mu^2 + 2) = \Sigma nm$. In a curve whose ordinate is $(\mu^2 - 1)/(\mu^2 + 2)$ and abscissa the frequency $p/2\pi$, all parts which lie outside the two horizontal dotted lines corresponding to ordinates $+ 1$ and $- \frac{1}{2}$ belong to regions of complete opacity; the points where the curve crosses the axis represent the free periods or bright lines. A mean continuous curve of dispersion may be sketched in, by a dotted line, which coincides with the actual curve in the parts where the dispersion is normal, and may be considered as gradually rising towards a band of intense absorption in the ultra-violet, which dominates the mean dispersion; near an absorption band the dispersion is anomalous, but if the band is narrow as in the case of gases, the anomaly is confined to very narrow range. The diagram here given represents a case of four free molecular periods, for the third of which g is negative while it is positive for the others. The refractive index that is determined by prismatic deviation is the real part of μ, taken positively (§ 34 *infra*). A graph of this quantity is represented by the thick broken line of the lower curve: thus near a free period $2\pi/p_1$ the ordinate rises to infinity when $p^2 = p_1^2 - ng_1$, then sinks instantly to zero, and remains zero until $p^2 = p_1^2 + 2ng_1$, when it becomes positive again. Slight general absorption would ease off the corners of this graph so



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various orientations which change from time to time owing to encounters. The effect of this is that the coefficients which represent the interaction between the aggregate of the matter and the æther, in the element of volume, are now the steady aggregates of the coefficients $c_1, c_2, \ldots\ldots c_n$ which belong to the various simultaneous orientations of the molecules. Thus the analysis remains intact provided $c_1, c_2, \ldots . c_n$ represent average values, and, where necessary, a coefficient of absorption is introduced to represent the abstraction of energy from the waves owing to the continual changes of molecular orientation. After each such change of orientation of a molecule, the energy of its previously accumulated synchronous vibration is radiated away or degraded into heat.

that it would not go up to infinity nor go down to zero: but there is nothing in its general aspect, at any rate for g positive, as for example given by VON HELMHOLTZ, and verified by PFLÜGER for anomalously dispersive solid dye stuffs, which specially favours any one theory of dispersion.

30. The medium has hitherto been taken as absolutely transparent, that is, no degradation of energy occurs in it, the absorption bands above so called being really produced by total and nearly total reflexion of the radiation, which thus is not absorbed by the medium, but simply cannot get into it. Suppose that there is present a conduction current, which may be considered to include all causes which put $\mathfrak{D}'$ out of phase with $\mathfrak{D}$ and so lead to regular absorption of the energy of the waves: it may be represented, as in Part II., § 11, by the formula

$$\mathfrak{C} = k(m' \, d/dt + \sigma')\,\mathfrak{D};$$

we now have

$$\mathrm{K}' = 1 + \{\mathfrak{D}' + (d/dt)^{-1}\,\mathfrak{C}\}/\mathfrak{D},$$

thus simply adding to the formula for the square of the index of refraction the terms $-(km' + kp^{-1}\sigma'\iota)/(m'^2p^2 + \sigma'^2)$, which satisfactorily represent the general features of metallic propagation as was shown in Part II., § 11.*

It is noteworthy that as the period becomes very rapid the effective index of refraction always approximates to unity; so that very short waves will not suffer sensible reflexion, refraction, or diffraction even while their length may include many molecules of the material medium. In fact when the period of the radiation is sufficiently high, the free periods of the polar molecules are not quick enough to enable them to respond,† while the comparatively free ions are prevented by their

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[* For example, it explains why the real part of μ^2 is negative for metals. In Part II. the generally received contention (based on a narrower theory) that μ^2 cannot be negative for purely elastic, that is dielectric, media was admitted without examination: but it is obviously inconsistent with the discussion above. As a concrete illustration, for a stretched thread weighted with equidistant particles, the square of the velocity of propagation of transverse waves of sufficiently short periods is negative: yet no inference follows as regards instability.]

† For a similar reason, the periods of luminous radiation are already too high to allow magnetic polarization to play any part in its propagation.

The statement in the text involves the currently received explanation of the Röntgen radiation. The different view has been recently advanced by Sir GEORGE STOKES that it may consist of sudden shocks transmitted through the æther from impacts by the molecules of the cathode streams. The molecules of matter lying in the track of the rays would not have time to be sensibly polarized by a sudden pulse which is over in a small fraction of their natural periods, and thus the pulse would pass across in the spaces between them, like sound through a grove of trees, without sensible refraction or diffraction: on the other hand the disruptive effect would resemble that of an explosive wave. Such pulses could hardly be other than the irregular beginnings of regular wave-trains sent out by the individual vibrating molecules; and as all radiation consists of such intermittent trains each with its irregular beginning, it would be assumed that the initial pulse is very much more intense in the electric bulb than in ordinary light, though still perhaps representing but a small portion of the total energy of the radiation. That the bombardment by the cathode streams is of a very disruptive, so to speak explosive, inertia from attaining any sensible velocity before the force is reversed; so that in neither way can the propagation of electric displacement across the medium be sensibly affected by the presence of the molecules. The formula shows, however, that the damping effect of conductivity usually persists to higher periods than the simple refracting effect of the excited vibrations.

The theory of dispersion would assume a simpler form if the molecules were systems vibrating about positions of equilibrium, instead of about states of steady motion. In that case the coefficients $e_{12}, e_{13}, \ldots . e_{1n}$ are null: the restoring forces, proportional to the velocities, to which these belong, are in fact introduced by the steady motions, and may be named, after Lord Kelvin, motional gyrostatic forces; they evidence themselves by causing slow precessional oscillations. The positional gyrostatic forces, or centrifugal forces proper, are merged in $\mathrm{T} - \mathrm{W}$ along with the forces arising from the potential energy.* When these motional forces are absent, we have $\theta_1 = c_1\mathrm{P}_1/(a_1 - \mathrm{A}_1 p^2)$ and similarly, so that $f_1/\mathrm{P}_1 = \Sigma c_1^2/(a_1 - \mathrm{A}_1 p^2)$; and as before, in a transparent medium, $(\mathrm{K}' - 1)/(\mathrm{K}' + 2) = \frac{4}{3}\pi \mathrm{C}^2 \mathrm{P}_1^{-1}\Sigma nf$. Thus the values of $g_1, g_2, \ldots . g_n$ are in this case all positive; so that, if this represented the facts, the fragments of a horizontal spectrum, with red on the left, would after further refraction by a prism of anomalous material with its edge horizontal and uppermost, all slope upwards from left to right. On the other hand, each change of sign from positive to negative among the successive values of $g_1, g_2, \ldots . g_n$ would give two fragments of the spectrum which would be curved back so as to be highest and lowest respectively near the middle, while a negative following a negative would imply slope upwards from right to left. According to Kundt's law the index is abnormally great on the lower side and abnormally small on the upper side of an absorption band; and if this generalization is universally valid, it will follow that $g_1, g_2, \ldots . g_n$ are actually all positive.

31. It is thus fundamentally desirable on various grounds to obtain information as to how the signs of these quantities depend on the gyrostatic coefficients; in particular because the present theory of gyrostatic molecules is a very wide one, and for example includes as a limiting case the hydrodynamical vortex atoms of Lord Kelvin, in which the constitution is purely gyrostatic but the number of degrees of freedom is infinite. It will be convenient for this purpose to change to new coordinates of which $c_1\theta_1 + c_2\theta_2 + \ldots . + c_n\theta_n$ is one, say the one with suffix unity: and to choose them semi-normal, so that the potential energy is represented by a

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character, compared with ordinary molecular encounters, is in keeping with the rapid disintegration and evaporation of metallic plates under its influence. It is conceivable that the long-continued Becquerel radiation from fatigued phosphorescent substances arises in like manner from very sudden release of their molecules into new groupings, in the course of their gradual return to a natural or unfatigued configuration.

* The periods of small free vibrations, and the amplitudes of small forced ones, would not be affected by reversal of *all* the gyrostatic momenta in the system: in fact this reversal would just change the system into its optical image.

sum of squares, which would be all necessarily positive if there were no gyrostatic influence. It may then be shown in the manner of the preceding analysis that f_1/P_1 is equal to a fraction of which the denominator is the period determinant of the molecule and the numerator is the minor of its leading term ; the numerator is therefore the period determinant of the same molecule when its leading coordinate is prevented by constraint from varying. Thus we have the theorem

$$f_1/P_1 = A_1^{-1}(p^2 - \alpha'^2_2)(p^2 - \alpha'^2_3) \dots (p^2 - \alpha'^2_n)/(p^2 - \alpha_1^2)(p^2 - \alpha_2^2) \dots (p^2 - \alpha_n^2),$$

where $(\alpha_1, \alpha_2, \dots \alpha_n)/2\pi$ are the natural frequencies of the molecule, while $(\alpha'_2, \alpha'_3, \dots \alpha'_n)/2\pi$ are its frequencies when it is subjected to that particular constraint (namely on a_1) which would prevent it from vibrating under the influence of the incident radiation ; also A_1 is the coefficient of inertia of that particular vibration, so that its kinetic energy is $\frac{1}{2}A_1\dot{\theta}_1^2$.

This constraint may be represented analytically by making the elastic coefficient a_1 infinite; we may therefore attempt to trace, by the examination of graphs of the separate terms involved, the effects on the free periods of continuously varying this constant. The behaviour of a gyrostatic system may be very different from what experience teaches as to vibrations about configuration of rest, for the mere imposition of constraint to limit the vibrations of one coordinate may upset the stability of others: thus if x represent p^2, the present period equation is of type $\phi(x) + a_1\psi(x) = 0$, in which all the n roots of $\phi(x)$ are real and positive, while the same may not be true of the $n - 1$ roots of $\psi(x)$. It follows, easily, however, by application of the principle of energy,* that if the system be completely stable when all the gyrostatic motional forces are removed, then it will remain stable when these forces are restored ; and stability will therefore also be maintained when elastic connexions are strengthened or constraints are introduced.† In that case the roots of $\psi(x)$ will all be real and positive : and it is easy to deduce that they will separate those of $\phi(x)$, and that in consequence $g_1, g_2, \dots g_n$ will be all positive, so that KUNDT's law will hold good. The further conclusion is thus also somewhat probable that if the constitution of the gyrostatic molecule is thoroughly stable so that the imposition of mere constraint could not upset it, then KUNDT's law will hold.

32. The specific refraction $(\mu^2 - 1)/(\mu^2 + 2)$ always increases along with the index μ : if the dispersion were controlled solely by powerful absorption bands in the ultra-violet, with positive g, the trend of the index would always be in the same direction as the frequency increases. Hence in the large class of substances with normal dispersion of visible light for which K exceeds μ^2, there must also be strong

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* *Cf.* THOMSON and TAIT's 'Nat. Phil.,' Ed. 2, Part I., p. 409 : the relations of the free periods of a gyrostatic system are there discussed at length in pp. 370–415.

† All the relations as to the march of the periods developed by RAYLEIGH ('Theory of Sound,' Ed. 2, §92*a*), from ROUTH's analysis will then hold good.

absorption in the ultra-red.[*] The specific refraction however always tends to the limit unity as μ^2 or K′ increases; so that the large dielectric constants of water and alcohol (at ordinary temperatures) are not so abnormal in their optical as in their electrical aspect. These large values are an indication that the constituents of the molecules are distantly and loosely connected together, which may be related to the powerful action of these substances as solvents;[†] it has been noticed that high inductive capacity is usually associated with conductivity.

33. It is of interest to contrast these results with the ones that flow from a purely mechanical theory of dispersion. If the molecule consist of a dynamical system, simple or gyrostatic, of dimensions small compared with the wave-length, joined on to the æther by mechanical connexions, the uniform oscillatory displacement of the æther will exert no differential statical force on the molecule, but the kinetic energy of the whole compound system will contain terms involving products of the

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[*] In VON HELMHOLTZ's memoir on the electric theory of dispersion, he found satisfactory agreement between the formula with one ultra-violet absorption band and the observations for glycerine, and he suggested that agreement might also be established for carbon bisulphide by assuming slight dissipation such as would not sensibly modify the laws of reflexion: but he apparently omitted to notice that this verification is defective in not making the square of the index equal to the statical dielectric constant for the case of very long waves. To amend it, another region of absorption would have to be assumed in the ultra-red, far down so as not to sensibly affect the visible radiations, thus leading to the KETTELER type of formula, which is approximately $\mu = A + B\lambda^2 + C\lambda^{-2}$ for slightly dispersive substances: in the case of glycerine C would be small. When there is only one absorption band, the dispersion formula common to these discussions is in form the same as one derived by VON LOMMEL ('Wied. Ann.,' 13, p. 353) from a mechanical theory, and compared by him with observation for a considerable number of media. In all media whose dispersion is effectively controlled by one absorption band, that band must be far in the ultra-violet or else the dispersion of the visible radiation will be excessive, so that the formula must approximately coincide with CAUCHY's: thus it is only substances for which μ^2 is approximately equal to but slightly greater than K, which can have any chance of coming into that class.

It appears from the above that in this formula for dispersion in a medium dominated by one main band of absorption, as given in Part II., §11, we must make the distinction, that the value of $2\pi/p$ for which μ^2 is infinite is not the free period of a single molecule by itself, or that of the bright line in a gaseous spectrum, but is the period when it is vibrating in step with all the surrounding molecules under whose influence it lies.

For very slow periods there is no dispersion in a transparent medium and the refraction depends wholly on the statical character of the medium, including its density. For higher periods of the incident radiation, the free periods of the molecules introduce dispersion and also absorption bands; but the position of these bands depends not merely on the free periods, but also slightly on the density of the medium, through the influence of the latter on its statical inductive capacity. It is not impossible that the free molecular periods, as well as the absorption bands, may be affected in this way. An influence of density of the medium on the position of the lines in the spectrum has been found and investigated by the Baltimore spectroscopists. [*Cf.* FITZGERALD, 'Astrophys. Journal,' 1896.]

[†] The reason here assigned is different from the one that has been given by various writers, that high inductive capacity of an intervening medium weakens the electric forces between the ions in the dissolved molecule. Here it is taken as an indication that the effective ions are far apart in the molecules of the solvent, so that a dissolved molecule can come under the influence of one of these ions alone, without much counteracting effect from the other ion.

displacement u of the æther and the coordinates $u, \theta_1, \theta_2, \ldots . \vartheta_n$ of the small disturbance of the molecule, say terms $(c_0\dot{u} + c_1\dot{\theta}_1 + c_2\dot{\theta}_2 + \ldots . + c_n\dot{\theta}_n)\,\dot{u}$. In the equation of propagation, formed in the LAGRANGIAN manner, $\rho\, d^2u/dt^2$ will now be replaced by $d^2/dt^2\,(\rho u + c_0 u + c_1\theta_1 + c_2\theta_2 + \ldots . + c_n\theta_n)$; while in the equation of vibration of the molecule terms of type $c_1\, d^2u/dt^2$ will occur. The square of the index of refraction is thus given by $\mu^2 = 1 + (c_0 u + c_1\theta_1 + c_2\theta_2 + \ldots . + c_n\theta_n)/\rho u$; and this leads by analysis similar to the above to a dispersion formula $\mu_2 = \mathrm{A} + \Sigma g_1/({p_1}^2 - p^2)$.* It is to be noticed that, on a mechanical theory, the index does not finally tend to unity as the frequency $p/2\pi$ rises, for when the waves have ceased to excite internal vibrations in the molecules the æther is still loaded by their inertia; an exception occurs when the attachment of the molecule to the æther is such that, when owing to the high period it is not internally vibrating, the æther does not sensibly displace its centre of mass, in which case the constant A is unity and there is no effective load on the æther. If we suppose that each molecule has an attachment to a very large mass, so as to be practically anchored to it in space, this will require us to take one of the natural frequencies to be infinite in the above analysis, so that say p_1 is zero. When both these characteristics are present, we arrive at LORD KELVIN'S formula.† If on the other hand we take the medium to be like an elastic jelly, permeated by spherical portions of different inertia and elasticity, the problem is a quite different one, which forms in fact a rude mechanical analogue of the electric theory; and it was in this way that L. LORENZ independently arrived at the specific refraction formula above discussed.

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* An equation equivalent to this with $g_1, g_2 \ldots .$ all positive, appears in SELLMEIER'S original paper ('Wied. Ann.,' 1872), based however on a much more special hypothesis.

† Baltimore Lectures, 1884: *cf.* also present memoir Part I., 'Phil. Trans.,' A, 1894, p. 820. In these lectures LORD KELVIN, with a view to explaining true absorption without introducing frictional forces into ultimate theory, contemplates the molecules as able to take up a vast amount of energy, near certain periods, before they attain to a steady state of synchronous vibration; as however that state must come after at any rate some millions of vibrations, and absorption would then cease, it is presumably part of the theory that the absorbed energy is constantly being degraded in the molecule by a process analogous to fluorescence, and so being got rid of by radiation at a lower period,—or it may be simply scattered owing to change of orientation of the steady state of vibration at which the molecule has arrived, due to encounters with other molecules, as indicated by JANSSEN'S law (*supra*) for gases. In the electric theory, metallic absorption is here taken to be chiefly due to the presence of free ions or electrons; but in weakly absorbing media it is probable that the former cause is the effective one. The only analytical way open for representing it is to introduce an absorption coefficient expressing the averaged rate at which the energy of the radiation is being exhausted.

The synchronously vibrating material molecules would not in any case give rise to further absorption by sending out energy in regular secondary waves: their uniformity in distribution and phase prevent this, just as they prevent the separate elements of the continuous æther from acting in the same way. The dust particles which give rise to the blue sky are irregularly distributed, and the individual secondary waves thence originating have irregular and independent phase-differences with reference to

Prismatic Deviation by Opaque Media.

34. The fact that light preserves its period shows (Part I., § 90) that the circumstances of its propagation across opaque media are determined simply by a complex index of refraction, of which the imaginary part represents the absorption. Measures of deviation by opaque prisms, such as those made by KUNDT, yield directly the value of this complex index, by simple consideration of the geometrical continuity of the traces of the waves along the interfaces, without the necessity of the intervention of any dynamical theory whatever and therefore free of all ambiguity of interpretation. The thickness of the portion of the prism that is traversed does not affect the deviation of the light; so it may be taken as null, and we have only to consider refraction into the prism, and then out of it at a plane inclined at an angle differing by α, the angle of the prism, without changing the point of incidence. Let the axis of x be in the first face of the prism, towards the edge, that of y normal to it and that of z parallel to the edge. Then for the incident, reflected, and refracted waves, of period $2\pi/p$, the vibration-vectors are proportional respectively to

$$\exp \iota (lx + my - pt), \ \mathrm{A} \exp \iota (lx - my - pt), \ \mathrm{A}' \exp \iota (lx + m'y - pt)$$

where

$$l^2 + m^2 = \mathrm{c}^{-2}p^2, \ l^2 + m'^2 = \mathrm{K}'\mathrm{c}^{-2}p^2,$$

c being the velocity in free space, and K′ the complex value of the square of the index of refraction. It will now be convenient to refer the second refraction to corresponding axes ξ, η, ζ related to the second face of the prism; thus

$$x = \xi \cos\alpha - \eta \sin\alpha, \ y = \xi \sin\alpha + \eta \cos\alpha, \ z = \zeta.$$

The vectors of the incident, reflected, and emergent beams are then proportional to

$$\mathrm{A}' \exp \iota (\lambda\xi + \mu\eta - pt), \ \mathrm{B} \exp \iota (\lambda\xi - \mu\eta - pt), \ \mathrm{B}' \exp \iota (\lambda\xi + \mu'\eta - pt),$$

where

$$\lambda = l \cos\alpha + m' \sin\alpha, \ \mu = -l \sin\alpha + m' \cos\alpha, \ \text{and} \ \lambda^2 + \mu'^2 = \mathrm{c}^{-2}p^2.$$

When the refracting angle α is small, this gives approximately $\mu'^2 = m^2 - 2\,lm'\alpha$,

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the primary exciting wave. The analogous medium for sound, filled with fixed attuned resonators, is absorbent solely on account of the secondary radiation of the resonators: consequently if they were all alike and regularly distributed and they occupied a very large number of wave-lengths, there would be no absorption and the medium would be transparent to sound, unless it has the same period as the resonators when it could not penetrate into the medium at all. The correlative absorption of light would thus be a process special to it, arising from ionisation and molecular impact. Unless the absorption in iodine vapour is accomplished by ionisation so that it goes mainly into heat, there must be scattered light accompanying it and representing part of it.

or $\mu' = m - l/m \, . \, m'\alpha$, where $m' = \{(l^2 + m^2) \mathrm{K}' - l^2\}^{\frac{1}{2}} = j + \iota k$ say. Thus the emergent vibration-vector is represented by

$$\mathrm{B}' \exp \iota \, \{(l + \alpha m') \, \xi + (m - l/m \, . \, \alpha m') \, \eta - pt\},$$

that is

$$\mathrm{B}' \exp - \alpha k \, (\xi - l/m \, . \, \eta) \exp \iota \, \{(l + \alpha j) \, \xi + (m - l/m \, . \, \alpha j) \, \eta - pt\}.$$

It therefore emerges at an angle of refraction ψ, away from the edge, given by

$$\cot \psi = (m - l/m \, . \, \alpha j) \, / \, (l + \alpha j) = m/l \, . \, \{1 - \alpha j l \, (l^{-2} + m^{-2})\},$$

the angle of incidence being ϕ where $\cot \phi = m/l$. Thus $\psi = \phi + j/m \, . \, \alpha$, and the deviation is $\psi - \phi - \alpha$ that is $\alpha \, (j' \sec \phi - 1)$, where j' is the real part of $(\mathrm{K}' - \sin^2 \phi)^{\frac{1}{2}}$. When the angle of incidence ϕ is small, the deviation is thus $(n - 1) \, \alpha$, where n is the real part of the complex refractive index $\mathrm{K}'^{\frac{1}{2}}$. Thus the experiments of KUNDT on metallic prisms, and of PFLÜGER on anomalously refracting media,[*] determine the march of n. Although $\sin^2 \phi$ is not usually very considerable compared with K′, and thus oblique incidence on the prism does not very greatly affect the deviation,[†] yet it would seem desirable to have observations at oblique incidence, as they would give data for determining the imaginary part of the index also by this uniform method, and thus its complete value. If this were known for the neighbourhood of an absorption band, we should possess all the data requisite to guide and correct theory in the matter of optical dispersion; but a knowledge of n by itself is not of much service in this respect. The value of this method of prismatic deviation lies in the fact that the complex index is determined without the intervention of any considerations as to dynamical theory or the effect of surface contamination on polarization, which must enter into the interpretation of experiments on reflexion.

The Mechanical Tractions on Dielectric Interfaces: and the Mechanical Bodily Forcive.

35. When the local part of the forcive on the polarized molecules of the medium, arising from their interaction with the neighbouring polarised molecules, is left out of account, the remainder, which is the mechanical force on the element of volume, is derived from the energy function $-(f'\mathrm{P} + g'\mathrm{Q} + h'\mathrm{R})$; this would be also a potential function of the forces were it not that in it only the electric force (P, Q, R) is to be varied. When however the dielectric is homogeneous, the negation of perpetual motions requires that $f'd\mathrm{P} + g'd\mathrm{Q} + h'd\mathrm{R}$ shall be a complete differential; thus when the law of induced polarization is linear, the force will be derived from a potential function $-\frac{1}{2}(f'\mathrm{P} + g'\mathrm{Q} + h'\mathrm{R})$, and so will be balanced, as regards the interior of the medium and as regards the translatory part, by a hydrostatic pressure

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[*] A. PFLÜGER, 'Wied. Ann.,' 56, 1895, p. 412.

[†] Cf. the measures of SHEA, 'Wied. Ann.,' 56.

$\frac{1}{2}(f'\mathrm{P} + g'\mathrm{Q} + h'\mathrm{R})$ + const.; and when the medium extends continuously to a distance from the seat of the electric action, the constant in this expression must be null. When the medium is isotropic, the translatory force is all, there being no torque on the element of volume. In a fluid dielectric this compensating hydrostatic pressure actually exists, and has been measured; in a solid it is merely a compendious expression for the material reaction per unit volume against the electric forces transmitted by the æther from other matter at a distance. If however the fluid dielectric is heterogeneous there will not be a potential function, and it can only be in equilibrium when stratified in a certain manner; if gravity did not operate the surfaces of stratification would be the equipotentials of the field of force.

36. When there are in the electric field interfaces of transition between different dielectrics, there will also exist surface-tractions on them which may be evaluated by considering an actual, somewhat abrupt, interface to be the limit of a rapid continuous variation of the properties of the medium which takes place across a layer of finite though insensible thickness. Let then the total displacement (f'', g'', h''), with its circuital characteristic where there is no free charge, be made up of the dielectric material polarization (f', g', h'), and the displacement proper (f, g, h) which is the æthereal elastic rotation $(\mathrm{P}, \mathrm{Q}, \mathrm{R})/4\pi$. Thus if we neglect now the minute difference between the æthereal force $(\mathrm{P}', \mathrm{Q}', \mathrm{R}')$ and the electric force $(\mathrm{P}, \mathrm{Q}, \mathrm{R})$,

$$df'/dx + dg'/dy + dh'/dz = -\rho', \qquad df/dx + dg/dy + dh/dz = \rho + \rho',$$

where ρ' is the POISSON ideal volume-density corresponding to the polarization, and ρ is the volume-density of free electrons, surface distributions being now by hypothesis non-existent.* The mechanical forcive acting in the dielectric is, per unit volume, a force $(\mathrm{X}', \mathrm{Y}', \mathrm{Z}')$ and a torque $(\mathrm{L}', \mathrm{M}', \mathrm{N}')$, where

$$\mathrm{X}' = f'd\mathrm{P}/dx + g'd\mathrm{P}/dy + h'd\mathrm{P}/dz + \rho\mathrm{P}, \qquad \mathrm{L}' = g'\mathrm{R} - h'\mathrm{Q}.$$

The component parallel to x of the aggregate force acting on the whole transitional layer is the value of $\int \mathrm{X}'d\tau$ integrated throughout it. This integral is finite, although the volume of integration is small, on account of the large values of the differential coefficients which occur in the expression for X. To evaluate it, we endeavour

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* The notation of Part II. is here maintained; thus (f'', g'', h'') represents the (f, g, h) of MAXWELL'S 'Treatise.' Electrostatic units are here employed. It may be well to recall the relations of these quantities. As the æthereal elemental rotation is from its nature circuital, the increment in its outward flux across any closed surface is equal to the amount of electrons that have crossed that surface into the enclosed region, arising partly from movement of free electrons, and partly from orientation of polar molecules over the surface so that one pole is inside and the other outside. Thus, (l, m, n) being the direction vector of the normal, and Δ representing a finite increment,

$$\Delta\int(lf + mg + nh)\,d\mathrm{S} = \Delta\int\rho\,d\tau - \Delta\int(lf' + mg' + nh')\,d\mathrm{S};$$

so that $\int(lf'' + mg'' + nh'')\,d\mathrm{S} = \int\rho\,d\tau$, which gives $df''/dx + dg''/dy + dh''/dz = \rho$.

by integration by parts to reduce the magnitude of the quantity that remains under the sign of volume integration, so that in the limit we may be able to neglect that part; thus we obtain $\int X' d\tau = \int (lf' + mg' + nh') \, P dS + \int (\rho' + \rho) \, P d\tau$. By the definition of electric force, (P, Q, R) is the force due to a volume distribution of density $\rho + \rho'$ and to extraneous causes; so that in the limit when the transitional layer is indefinitely thin, we have, by COULOMB's principle, $\int (\rho + \rho') \, P d\tau = \frac{1}{2} \int (\sigma' + \sigma) (P_2 + P_1) \, dS = (8\pi)^{-1} \int (N_2 - N_1) (P_2 + P_1) \, dS$, P_1, P_2 being the values of the x component P, and N_1, N_2 those of the normal component N, of the electric force (P, Q, R) on the two sides of the layer, all measured towards the side 2, while σ' and σ are the surface densities constituted in the limit by the aggregates of ρ' and ρ respectively taken throughout the layer. Hence in the limit $\int X' d\tau = \Big| \int (lf' + mg' + nh') \, P dS \Big|_1^2 + (8\pi)^{-1} \int (N_2 - N_1) (P_2 + P_1) \, dS$. Thus the electric traction on the interface of transition may be represented by a pull towards each side, along the direction of the resultant electric force F; this pull is on the side 2 of intensity $n'_2 F_2 - \frac{1}{2} (n'_2 - n'_1 - \sigma) F_2$, that is $\frac{1}{2} (\sigma + n'_2 + n'_1) F_2$ in the direction of F_2, where n' is the normal component of the polarization of the medium measured positive towards the side 2; on the face 1 the pull is $\frac{1}{2} (\sigma - n'_2 - n'_1) F_1$ now in the direction of F_1, n' being measured positive as before. As the tangential component of the electric force F is under all circumstances continuous across the interface, the total traction on both sides combined is along the normal, and equivalent to $\frac{1}{2} (n'_2 + n'_1) (N_2 - N_1)$ together with the tractions $\frac{1}{2} \sigma F_2$, $\frac{1}{2} \sigma F_1$ acting on the true charge σ, all the quantities being now measured positive in any the same direction. If n'' denote the normal component of the total displacement (f'', g'', h''), so that $n'' = N/4\pi + n'$, $n''_2 - n''_1 = \sigma$, the first part of this total traction is $\frac{1}{2} (n''_2 + n''_1 - N_2/4\pi - N_1/4\pi) (N_2 - N_1)$, which is simply $- 2\pi n'^2_2 + 2\pi n'^2_1$ towards the side 2.* When the interface is between a dielectric 1 and a conductor 2, the traction is only towards the side 1 and is equal to $\frac{1}{2} (n'_1 + \sigma) F_1$, or $\frac{1}{2} n''_1 F_1$, per unit area, along the normal which is now the direction of the resultant force.

All this is quite independent of the law of the connexion between the polarization and the electric force in the material medium. Thus, under the most general circumstances as regards electric field, whether there is material equilibrium or not, the forcive on the material due to its electric excitation consists of the interfacial tractions thus specified, together with a force (X′, Y′, Z′) and a torque (L′, M′, N′) per unit volume, given by the formulæ $(X', Y', Z') = (d/dx, d/dy, d/dz) (f'P + g'Q + h'R)$ and $(L', M', N') = (g'R - h'Q, \; h'P - f'R, \; f'Q - g'P)$, in the former of which (f', g', h') is not to be differentiated.

The assumption underlying this analysis, that the transitions are gradual, will be

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* It may be recalled that in the terminology here employed, the *true* electrification σ is the density of unpaired electrons; while the *true* electric current arises from the movements of all the electrons, free and paired, but does not include the change of æthereal strain which must be added in order to make up the *total* circuital current of MAXWELL.

sufficiently satisfied even if the intermediate layer is only one or two molecules in thickness; for as these molecules are arranged slightly in and out, and not in exact rows along the interface, their polarity can still be averaged into a continuous density as above. The aggregate tractions over a thin layer of transition thus do not depend sensibly on the nature of the transition, but only on the circumstances on the two sides of the layer.

37. In the case of a fluid medium, the bodily part of the forcive produces and is compensated by a fluid pressure $\int i' d\mathrm{F}$, where i', being the polarization induced by the electric force F, is for a fluid in the same direction as F and a function of its magnitude. This pressure will be transmitted statically in the fluid to the interfaces;* combining it there with the surface traction proper, it appears that the material equilibrium of fluid media is secured as regards forces of electric origin if extraneous force is provided to compensate a total normal traction towards each side of each interface, of intensity $-2\pi n'^2 - \int i' d\mathrm{F}$. In the case usually treated, in which a linear law of induction is assumed so that the relation between i' and F is $i' = (\mathrm{K}-1)\,\mathrm{F}/4\pi$, the mechanical result of the electric excitation of the fluid medium is easily shown to be the same† as if each interface were pulled towards each side by a FARADAY-MAXWELL stress, made up of a pull $\mathrm{KF}^2/8\pi$ along the lines of force and an equal pressure in all directions at right angles to them. But this imposed geometrical self-equilibrating stress-system would not be an adequate representation of the mechanical forcive in a solid medium; for then the bodily forcive, instead of being wholly transmitted, is in part balanced on the spot by reactions depending on the solidity of the material. The forcive acting on isotropic material may however in every case, whether the induction follows a linear law or not, be expressed as an extraneous or imposed system, made up of bodily hydrostatic pressure $\int i' d\mathrm{F}$ (which in the case of a fluid only relieves the ordinary fluid pressure and so diminishes the compression, § 79 *infra*) together with normal tractions on the interfaces between dielectric media, of intensity $-2\pi n'^2 - \int i' d\mathrm{F}$ acting towards each side, and tractions $\frac{1}{2} n'' \mathrm{F} - \int i' d\mathrm{F}$ on the surfaces of conductors acting towards the dielectric.

38. A similar analysis applies to the electromagnetic forcive acting on a magnetically polarized medium. Excluding as before the part arising wholly from the interaction of neighbouring molecules, which (§ 44 *infra*) is not transmitted by material stress, but is compensated on the spot by molecular action due to change of physical state induced by it, the electromagnetic forcive proper is made up of a bodily force (X, Y, Z) and torque (L, M, N), where, (u', v', w') representing the true current,

$$\begin{aligned}\mathrm{X} &= v'\gamma - w'\beta + \mathrm{A}\, d\alpha/dx + \mathrm{B}\, d\alpha/dy + \mathrm{C}\, d\alpha/dz \\ &= vc - wb + \mathrm{A}\, d\alpha/dx + \mathrm{B}\, d\beta/dx + \mathrm{C}\, d\gamma/dx - \gamma dg/dt + \beta dh/dt,\end{aligned}$$

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* That is, a reacting pressure $\int i' d\mathrm{F}$ exerted by the interface will keep the medium in internal equilibrium: no constant term is added because the pressure must vanish along with the polarization.

† It is a normal traction equal to $-(\mathrm{K}-1)(\mathrm{KN}^2 + \mathrm{T}^2)/8\pi$ towards each medium, or in all a single traction $(\mathrm{K}_2 - \mathrm{K}_1)(2\pi n''^2/\mathrm{K}_1\mathrm{K}_2 - \mathrm{T}^2/8\pi)$ towards the medium 1.

and $L = B\gamma - C\beta$. Under the usual circumstances, in which the æthereal displacement-current can be neglected, these expressions are identical with the ones given without valid demonstration in MAXWELL's 'Treatise.'* The remarkable property is there established (*loc. cit.*, § 643) that, independently of the form of the relation between magnetic induction and magnetic force in the medium and whether there is permanent magnetism or not, this bodily forcive (with the last terms neglected) can be formally represented in explicit terms as equivalent to an imposed stress: viz. $\mathfrak{H}$ denoting magnetic force and $\mathfrak{B}$ magnetic induction, the bodily forcive is the same as would arise from (i) a hydrostatic pressure $\mathfrak{H}^2/8\pi$, (ii) a tension along the bisector of the angle ϵ between $\mathfrak{H}$ and $\mathfrak{B}$, equal to $\mathfrak{H}\mathfrak{B}\cos^2\epsilon/4\pi$, (iii) a pressure along the bisector of the supplementary angle, equal to $\mathfrak{H}\mathfrak{B}\sin^2\epsilon/4\pi$, together with an outstanding bodily torque turning from $\mathfrak{B}$ towards $\mathfrak{H}$ and equal to $\mathfrak{H}\mathfrak{B}\sin 2\epsilon/4\pi$. When $\mathfrak{B}$ and $\mathfrak{H}$ are in the same direction, the torque vanishes, and a pure stress remains in the form of a tension $(\mathfrak{H}\mathfrak{B} - \frac{1}{2}\mathfrak{H}^2)/4\pi$ along the lines of force and a pressure $\mathfrak{H}^2/4\pi$ in all directions at right angles to them. There is no warrant for taking this stress to be other than a mere geometrical representation of the bodily forcive. It is however a convenient one for some purposes.† Thus the traction acting on the layer of transition between two media, in which (α, β, γ) changes very rapidly, which might be directly deduced in the same manner as the electric traction above (§ 35), may also be expressed directly as the resultant of these MAXWELLIAN tractions

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* Vol. II., § 640. It will be observed that the force acting on the moving electrons which constitute the true current is here taken to be $(v'\gamma - w'\beta, \ldots, \ldots)$. In the investigation of Part II., § 15, which determines the motional force on a single electron, the expression for T represents the kinetic energy of the æther; it is transformed so as to be expressed in terms of the electric displacement of the æther and the electrons of the materials by introducing (F, G, H) whose curl gives the actual velocity of the æther near the electron; and finally, after the forces acting on the electrons and on the æthereal displacement have thus been separated out, (F, G, H) is eliminated by the same relation. Thus the force acting on the single moving electron comes out as $e(\dot{y}\dot{\zeta} - \dot{z}\dot{\eta} - \dot{F}, \ldots, \ldots)$, where $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ is the velocity of the medium; and the average force acting on the electrons in the element of volume, that is, the induced electric force causing electric separation in the element, is $e(\dot{y}c - \dot{z}b - \dot{F}, \ldots, \ldots)$, as there given. But in computing, as in Part II., § 23, the electromagnetic force on an element of volume carrying a current, it must be borne in mind that part of the above force on the single electron arises from the magnetism in this element of volume itself; and the principle of energy forbids that any part of the forcive on the mechanical element of volume of the material can arise from mutual actions inside the element, so that this part must be compensated by a reciprocal action of the moving electrons which constitute the current on those which constitute the magnetism, in a manner which might be expressed if necessary. Hence, when this local part is omitted in accordance with the general principle, the transmitted electromagnetic force is $(v'\gamma - w'\beta, \ldots, \ldots)$ as above, not $(v'c - w'b, \ldots, \ldots)$ as previously stated in closer accordance with the AMPÈRE-MAXWELL formula. *Cf.* § 44 *infra*. [Observe, however, that in quoting Part II., § 15 $(\dot{\xi}, \dot{\eta}, \dot{\zeta})$ must now represent the velocity of the æther multiplied by the square root of 4π times its very high coefficient of inertia: the unit of time was there tacitly chosen so that this factor should be unity.]

[† For example, the repulsion exerted by alternating currents on pieces of copper or other conducting masses may thus most conveniently be represented.]

towards the two sides of the interface. As there cannot be free magnetic surface-density or purely superficial current-sheets, the traction on the interface is represented, under the most general circumstances, whatever extraneous magnetic field may there exist, by purely normal pull of intensity $2\pi\nu^2$ towards each side, where ν is the normal component of the magnetization at that side. When the medium is non-magnetic, there is no such superficial traction, but only the bodily electromagnetic forcive on the *true* electric currents of the material medium, which is represented by the above stress system.

39. The form of the mechanical forcive is identical whether the polarization is electric or magnetic, provided there are no electric currents; in the first case it is the material reaction to the static strain in the æther, in the other case it is the reaction to the motional æthereal forcive arising from the revolving electrons in the molecule. Omitting for simplicity the slight effect of the convection current in cases where any exists, the forcive arising from the electric polarization of the medium consists of a bodily force (X′, Y′, Z′) and torque (L′, M′, N′), where

$$X' = f' \frac{dP}{dx} + g' \frac{dQ}{dx} + h' \frac{dR}{dx} + \rho P + g' \frac{dc}{dt} - h' \frac{db}{dt}, \quad L' = g'R - h'Q,$$

together with an interfacial traction between media 1 and 2 which is along the normal and equal to $-2\pi n'^2_2 + 2\pi n'^2_1$ towards the medium 2, n' representing the component of the polarity (f', g', h') along the normal; the motional forcive arising from the magnetic polarity and the electric currents, consists of a bodily force (X, Y, Z) and torque (L, M, N) where

$$X = A \frac{d\alpha}{dx} + B \frac{d\beta}{dx} + C \frac{d\gamma}{dx} + vc - wb - \gamma \frac{dg}{dt} + \beta \frac{dh}{dt}, \quad L = Bc - Cb,$$

together with a normal interfacial traction $-2\pi\nu_2^2 + 2\pi\nu_1^2$ towards the medium 2, ν representing the normal component of the magnetization (A, B, C). When the æthereal displacement current is neglected, the latter forcive is the same as would arise from MAXWELL'S magnetic stress specification. It may be shown that $X' = d/dx\,\{f''P - \frac{1}{2}(fP + gQ + hR)\} + d/dy\,(g''P) + d/dz\,(h''P) + hdb/dt - gdc/dt$, so that the former forcive is what would arise from an analogous electric stress specification in which (P, Q, R), (f', g', h'), $4\pi\,(f'', g'', h'')$, correspond to (α, β, γ), (A, B, C), (a, b, c) respectively, with the exception however in this case also of an outstanding bodily forcive $(hdb/dt - gdc/dt, \ldots.,\ldots.)$ which is not included in the stress. A theory which assumes that there is but one medium in which everything is transmitted by contact action, not two interacting media matter and æther as here, is compelled to get rid of any outstanding forcive like this, which is not expressible explicitly in terms of stress: for this reason supporters of that view have found it necessary to introduce into the electric field a purely hypothetical mechanical forcive arising from the electric field acting on the so-called magnetic current $d/dt\ (a, b, c)$, in analogy with the AMPEREAN forcive arising from the magnetic field acting on the electric current. The addition of this forcive $(hdb/dt - gdc/dt, \ldots, \ldots)$ to (X', Y', Z') and the omission of $(-\gamma dg/dt + \beta dh/dt, \ldots, \ldots)$ from (X, Y, Z), permits both to be expressed *explicitly* in terms of stress.

MAXWELL'S *Theorem of a Representative Stress.*

40. The mechanical forcive acting in a polarized medium thus corresponds in the main to the system of bodily force and interfacial traction which is the result of MAXWELL'S magnetic stress ('Treatise,' § 640) considered as an extraneous system applied to the medium. The electric stress of MAXWELL ('Treatise,' § 105) is something wholly different, leading in the case of homogeneous media to interfacial tractions only, without bodily force; it could thus have valid application only to unpolarized media, as for example to the theory of gravitation which passes through material bodies just as through a vacuum. The proposition really established[*] is that the mechanical forcive due to attraction at a distance, obeying the law of inverse squares, between material bodies, may be represented by a connexion in the form of an imposed extraneous stress symmetrical with respect to the lines of force, acting across the intervening medium, *provided* that medium is not in any way polarized by the force. A stress restricted by this relation of symmetry involves only two variables, the principal tractions along and at right angles to the line of force; and the essence of MAXWELL'S theorem is that it is possible always to determine these two variables so as to satisfy the three equations of equilibrium of the element of volume of the medium. These principal tractions prove, as is well known, to be equal in magnitude but opposite in sign. The proposition is in itself so remarkable that it deserves to be formulated abstractly without reference to hypothetical applications. The representation of a given bodily forcive by a geometrical stress-system is in general a widely indeterminate problem, as the six stress components have to satisfy only three equations: but the condition of symmetry with respect to lines of force restricts the stress so much that such a representation would only in special circumstances be possible.

The regular local Molecular Forcive in an excited Dielectric: its Expression as a Stress-system: Examples of the Principle of the Mutual Compensation of local Molecular Forcives.

41. In the above estimate of the mechanical forces acting on an element of a polarized medium, the influence of the general mass of the medium on the molecules in the element has been alone included; it remains to consider the *rôle* of such terms as would arise from the special forcives of neighbouring molecules. The intensity of

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[*] MAXWELL, "On physical lines of force," Part I., 'Phil. Mag.' 21, 1861, especially Prop. III.

the local part of the regular electric force acting at a molecule has already been assigned (§ 19) as $\tfrac{4}{3}\pi i'$, very approximately for the case of fluid media, possibly not so approximately for solids. The argument was that owing to the translational mobility of the surrounding molecules, their action on the one under consideration averages into that of the uncompensated distribution of poles which would exist on the surface of a small spherical cavity in a continuous uniformly polarized medium,—or, more precisely, into that of a spherical shell of poles, of thickness not indefinitely small but with this law of distribution around the centre. For the interior of a uniformly polarized medium the local part of the electric force is thus at each instant constant throughout this cavity and equal to $\tfrac{4}{3}\pi i'$; therefore the mechanical force exerted on the polar molecule (that is one involving equal numbers of positive and negative electrons) at the centre of the cavity is null, as it depends on the rate of variation of this electric force. But at a place where the polarization varies from point to point, the alteration in the law of surface-density over the cavity will supply a local part.

When the polarization i' changes only in magnitude and not in direction, this part will arise from a distribution of uncompensated poles over the surface of the cavity, of density $-(i'_0 + x\,di'_0/dx + y\,di'_0/dy + z\,di'_0/dz)\cos\theta$, where the subscript zero implies the value at the centre. If the axis of x is taken along the direction of i'_0, the electric potential U in the interior due to this distribution is equal to

$$-\tfrac{4}{3}\pi\left(i'_0 + \tfrac{1}{3}\frac{di'_0}{dx}\right)x - \tfrac{4}{5}\pi\left\{\tfrac{1}{3}\frac{di'_0}{dx}(2x^2 - y^2 - z^2) + \frac{di'_0}{dy}xy + \frac{di'_0}{dz}xz\right\}.$$

On a molecule of moment μ_x, at the centre, this gives a force

$$-\mu_x d/dx\,(d/dx,\ d/dy,\ d/dz)\,\mathrm{U},\ \text{that is}\ \tfrac{4}{5}\pi\mu_x\,(\tfrac{4}{3}d/dx,\ d/dy,\ d/dz)\,i'_0.$$

Thus there is a bodily force due to this cause, of intensity $\tfrac{2}{5}\pi\,(\tfrac{4}{3}d/dx,\ d/dy,\ d/dz)\,i'^2_0$; but there is not any bodily torque.

42. Now let us proceed to the general case, in which the direction of the polarization (f', g', h'), as well as its magnitude, varies from point to point; in the hypothetical case in which the effective distance between the poles of a molecule is small compared with the average distance between neighbouring molecules, we can express the molecular part of the forcive on an element of volume by simple summation for f', g', and h' separately, by aid of the expressions just found. Thus it consists of a bodily force $(\mathrm{X}_1, \mathrm{Y}_1, \mathrm{Z}_1)$ and torque $(\mathrm{L}_1, \mathrm{M}_1, \mathrm{N}_1)$, where $\mathrm{X}_1 = f'd\mathrm{P}_1/dx + g'd\mathrm{P}_1/dy + h'd\mathrm{P}_1/dx$, $\mathrm{L}_1 = g'\mathrm{R}_1 - h'\mathrm{Q}_1$, $(\mathrm{P}_1, \mathrm{Q}_1, \mathrm{R}_1)$ being the *local* part of the electric force in the spherical cavity, so that

$$\mathrm{P}_1 = \tfrac{4}{5}\pi\left\{\left(\tfrac{4}{3}x\frac{df'}{dx} + y\frac{df'}{dy} + z\frac{df'}{dz}\right) + \left(y\frac{dg'}{dx} - \tfrac{2}{3}x\frac{dg'}{dy}\right) + \left(z\frac{dh'}{dx} - \tfrac{2}{3}x\frac{dh'}{dz}\right)\right\}.$$

Hence

$$\mathrm{X}_1 = \tfrac{4}{5}\pi\left\{\left(f'\frac{d}{dx} + g'\frac{d}{dy} + h'\frac{d}{dz}\right)f' + \tfrac{1}{2}\frac{d}{dx}(f'^2 + g'^2 + h'^2) - \tfrac{2}{3}f'\left(\frac{df'}{dx} + \frac{dg'}{dy} + \frac{dh'}{dz}\right)\right\}$$

$$= \tfrac{4}{5}\pi\left\{\frac{d}{dx}(f'^2 + g'^2 + h'^2) - \tfrac{2}{3}f'\left(\frac{df'}{dx} + \frac{dg'}{dy} + \frac{dh'}{dz}\right) - g'\left(\frac{dg'}{dx} - \frac{df'}{dy}\right) + h'\left(\frac{df'}{dz} - \frac{dh'}{dx}\right)\right\},$$

with similar expressions for Y_1 and Z_1; while the torque vanishes in the limit.

43. In these formulæ the aim has been simply to represent as they are the regular local forcives acting on the molecules, as a distribution of force throughout the volume and, if need be, of traction over the surfaces of the material, thus avoiding the use of any hypothetical stress-system which might be a geometrical equivalent. It will presently be shown that an extension of the ideas underlying the YOUNG-POISSON principle of the mutual compensation of molecular forcives, employed in the theory of capillary action, requires that this local forcive shall set up a purely local physical disturbance of the molecular configuration in the material, until it is thereby balanced; in the case of an isotropic medium in a steady state it must thus necessarily be expressible as an imposed stress symmetrical with respect to the direction of polarization.

Let us, therefore, with a view to the verification of this proposition, analyze the effects of an internal stress symmetrical with respect to the lines of some kind of polarization denoted generally by i or (f, g, h). Such a stress must be of the type of a tension $(p + q)\,i^2$ along these lines combined with a tension qi^2 in all directions at right angles to them; for the stresses we are examining clearly vary as the square of the polarization. Thus the stress must be made up of a hydrostatic pressure $-qi^2$ combined with a tension pi^2 along the lines of the polarization. The tractions exerted by the latter part on elements of interface parallel to the coordinate planes yz, zx, xy are, per unit area, (qf^2, qfg, qfh), (qgf, qg^2, qgh) and (qhf, qhg, qh^2). Hence the total force exerted by the stress on the element of volume $\delta x\,\delta y\,\delta z$ is, per unit volume, (X, Y, Z) where

$$\mathrm{X} = \frac{d}{dx}(pi^2) + \frac{d}{dx}(qf^2) + \frac{d}{dy}(qgf) + \frac{d}{dz}(qhf)$$

$$= (p + \tfrac{1}{2}q)\frac{d}{dx}i^2 + qf\left(\frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}\right) + qg\left(\frac{df}{dy} - \frac{dg}{dx}\right) + qh\left(\frac{df}{dz} - \frac{dh}{dx}\right);$$

and, the stress being self-conjugate, there is no torque. On comparison of this force with the local molecular, or cohesive, force on the element of volume, of electric origin, expressed above, it appears that they are of the same type provided $f\,dx + g\,dy + h\,dz$ is an exact differential, which is the case with the equilibrium electric polarization i' or (f', g', h') induced in an isotropic medium, the electric force being always under conditions of equilibrium circuital. The material stress which represents the regular electrostatic part of the molecular forcive by which the molecules hang together, is therefore a tension $\tfrac{2}{3}.\tfrac{4}{5}\pi i'^2$ along the lines of the polarization i' combined with an equal pressure $\tfrac{2}{3}.\tfrac{4}{5}\pi i'^2$ in all directions at right angles to them.* If, however, the medium were crystalline, the stress would be of a more complex type than this, being related to the crystalline axes as well as the axis of polarization. When the interface is the surface of a conductor, the forcive on the charge of free electrons which pervades the layer of transition adds nothing to this effect beyond what has been already set down ; for the electric force due to a volume-distribution of single poles or electrons has no finite part depending solely on the element of volume at which its value is expressed, that is, it involves no molecular term.

44. The analysis here given is not however numerically applicable to a case in which the effective distance between the poles of a molecule is comparable to the distance between neighbouring molecules. The system formed by a bundle of iron nails suspended from the pole of a magnet and hanging on to each other against gravity, which has been used as an illustration of the molecular part of the forcive in the previous papers, does not come under these formulæ. That system may however be employed with advantage as a real illustration of the general principles, especially if we imagine the magnetized iron nails to be connected by springs or imbedded in an elastic matrix. When no extraneous forces such as gravity act on this model of a molecular medium, it adjusts itself into a condition of internal equilibrium, in which attractions between the magnetic nails are locally balanced by repulsions exerted by the springs. The various local molecular forcives, typified here by these attractions between magnets and forces exerted by springs, precisely compensate each other in each portion of the medium. If an additional magnetic field is introduced, which alters the magnetic polarities of the nails, the parts of the medium will change their shapes and volumes until compensation again supervenes : there will thus occur an intrinsic deformation of the medium, and there may be also intrinsic changes of its physical properties, associated with the polarization and proportional in simple cases to its square. Suppose now that an extraneous force like gravity, or the magnetic field arising from the medium as a whole, begins to act, that is, a regular mechanical force on the medium in bulk so that it is in the aggregate proportional to the volume on which it acts ; this will produce a further deformation, but one proportional to the first power of the exciting force. The local internal molecular forcive will again no longer be exactly balanced ; but the unbalanced part will possess at each point the characteristics of an elastic stress system, because when the element of volume is small enough the tractions thus arising over its surface must equilibrate without any assistance from the then negligibly small extraneous bodily forcive. Even then however this elastic stress excited by an external field cannot be specified in terms of surface tractions unless the dimensions of the smallest element of volume which the circumstances require us to consider are large compared with the range of the intermolecular forces. Unless that is the case, the energy of elastic strain of the element of the medium, expressed in GREEN's manner, will involve higher fluxions of

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* This is an example of MAXWELL's theorem, § 40 *supra*.

the displacement in addition to those of the first order, and the equilibrium between two contiguous portions will not depend on continuity of displacement and of surface traction alone : other quantities also would have to be continuous for which there is no interpretation in the ordinary analysis of elastic reactions : the elastic stress would in fact not then be expressible in terms of tractions on interfaces. In such a case the only procedure that seems open, as the science of mechanics is now constituted, would be to transfer the effects of that part of the elastic energy which involves higher differential coefficients to the class of intrinsic or non-mechanical deformations.

45. In this theory of electric polarization the division of the forcive per unit volume into a molar and a molecular part has been made by means of the ideal volume and surface densities of POISSON, which are the equivalent as regards outside points of the actual polarization of the material. This method consists essentially in computing the forcive by combining opposed poles of neighbouring elements, instead of taking the single polarized element as the unit ; it shows that these adjacent poles nearly compensate each other except as regards a simple volume density whose attraction has no molecular part, and a surface density partly at the outer surface and partly at the surface of the cavity which contains the point under consideration. The effect of the latter surface density, depending as it does wholly on the immediate surroundings, is the molecular or cohesive part of the average forcive.

These principles may be enforced and illustrated by contrast with a procedure by separate molecules which would usually lead to a different result; it will suffice to consider the case in which the polarization is uniform in direction throughout the material. If the axis of x be taken in the direction in which the intensity of the electric polarization changes most rapidly in the neighbourhood of the point considered, it is easy to see that the bodily force on an element due to the surrounding polar molecules is parallel to x and equal to $-(\mathfrak{a}f'\,df'/dx+\mathfrak{b}g'\,dg'/dx+\mathfrak{b}h'\,dh'/dx)$, and thus derivable from a potential function $-\frac{1}{2}(\mathfrak{a}f'^2+\mathfrak{b}g'^2+\mathfrak{b}h'^2)$, where $\mathfrak{a}$, $\mathfrak{b}$ are constants. In the case of an interface of rapid transition from one uniformly polarized medium to another, there is thus a forcive only in the transition layer, and its integral throughout that layer is equivalent to a traction parallel to the axis of x, of intensity $-\frac{1}{2}(\mathfrak{a}f'^2+\mathfrak{b}g'^2+\mathfrak{b}h'^2)\sin^2\theta$ where θ is the angle between the interface and the axis of x, pulling at the interface into each medium. If the polarization (f', g', h'), or i', is normal to the interface, this traction is $-\frac{1}{2}\mathfrak{a}i'^2\sin^2\theta$, if tangential it is $-\frac{1}{2}\mathfrak{b}i'^2\sin^2\theta$. To estimate the values of $\mathfrak{a}$ and $\mathfrak{b}$, we may consider separately the forcives exerted by the molecules of the polarized medium on μ_x, μ_y, μ_z, the components of a molecular moment μ situated in the neighbourhood of the interface, in the case when the interface is normal to the axis of x. The outstanding terms in the aggregate forcive due to the surrounding molecules, which do not cancel each other by symmetry, are normal to the interface and make up $\mu_x\Sigma\mu'_x\,d^3r^{-1}/dx^3+\mu_y\Sigma\mu'_y\,d^3r^{-1}/dx\,dy^2+\mu_z\Sigma\mu'_z\,d^3r^{-1}/dx\,dz^2$; or, per unit volume, $-\frac{1}{2}(\mathfrak{a}f'^2+\mathfrak{b}g'^2+\mathfrak{b}h'^2)$ wherein $\mathfrak{a}=-2\mathfrak{b}$ because $\nabla^2r^{-1}=0$.

The point, however, to be noticed is that this expression for the *total* traction on an interface, due to both molar and molecular forcive, which makes the pull on an interface lying normal to the lines of polarization twice as great as the push on one tangential to those lines, holds only for the case in which the material is polarized in the same direction throughout the whole extent of its volume. We may estimate by itself the action of the surrounding portion only, extending to any distance we please; but the action of the remaining outside part of the medium will still involve that of an inner surface density of uncompensated poles which will remain of undiminished order of magnitude. This procedure by separate molecules is thus not suitable for discrimination between the forcive due to the medium as a whole, which is transmitted, and the molecular forcive which is compensated locally.

46. The justification of the theory here applied, which balances on the spot the molecular part of the forcive due to the electric polarization, by an intrinsic cohesive stress in the material which is independent of the material elastic constants and strains at the place, may be further enforced by consideration of the ideally simple case of a gas. If a system of bodily forces act on it from a distance, they can always be balanced by a simple increase of pressure when they are derived from a potential function; while if they were not so derived the medium could not be in equilibrium. The dual phenomenon of equilibrium of the element of volume maintained by a balance between two forcives, an extraneous and an internal one, is really a balance between a forcive on the element of volume acting from a distance by the mediation of the æther, and another forcive arising according to the explanations of the theory of gases from the impacts of the molecules surrounding the element and, in the case of dense media, in part also from cohesive molecular actions. It is in this case a balance between a static bodily forcive and a steady kinetic molecular one; if the force transmitted through the æther from a distance increases, and equilibrium is to be maintained, the molecular configuration must be adjusted so that the impacts and the local molecular attractions shall continue to preserve the balance. When to the forces acting from a distance are added coordinated electric attractions between the molecules of the polarized medium, a further adjustment of molecular configuration must ensue. Now when the gas is electrically polarized, the attractions between neighbouring molecules give a forcive, not isotropic like a fluid pressure, but depending on the direction of polarization; its action will thus alter the originally fortuitous arrangement of the velocities of the molecules of the gas so as to impart to their distribution a slightly axial character, and when this has resulted in a new steady state the pressure due to the impacts will be different according to the manner in which the element of interface that is pressed is related to the line of polarization. This molecular addition of an intrinsic local stress, which has not the character of the ordinary fluid pressure, will just balance the action of the local electric attractions when the state of the system has again become steady; and being thus itself completely compensated *locally*, there will remain nothing of the molecular part of the

electric mechanical forcive to be transmitted across the material medium. The argument applies with suitable modification to any isotropic medium, as well as to a gas; for an æolotropic solid the specification of the actual molecular stress of different origin which thus balances for each element the molecular electric forcive will be more complicated, involving the axes of æolotropy as well as the axis of polarization.

47. For present purposes the important consequence is that, under circumstances of equilibrium, that part of the forcive on an element of a material body which arises from the excitation of neighbouring molecules and is expressed in terms of them alone, is not transmitted by material stress, but forms a balance on the spot with the cognate internal molecular forcives of other types.* The only circumstance that might apparently vitiate this conclusion would be that the transitions between different media may be too abrupt to be treated, from the point of view of individual molecules, as really gradual transitions, after the manner of the above analysis; but even if we could imagine such a case, the discrepancy must for fluids be made up, provided the interface is a permanent one, by capillary forces in the interfacial layer, the effect of an outstanding surface derangement of energy.

In a dielectric body situated in an electric field there is thus the mechanical strain due to the field; and there are also intrinsic change of volume and other dimensions and of physical properties, proportional to the square of the local polarization. If the dielectric is solid, those changes of dimensions may not fit in with the continuity of the material without the intervention of secondary strains; but in fluid media the case is simple and precise, as no strain other than mere compression can exist.

The Mutual Compensation of Local Molecular Agencies: Organized and Unorganized Energy: The Single Postulate of Thermodynamics, Available and Degraded Energy: Physical Basis of the Idea of Temperature.

48. The scope of these molecular considerations (§§ 43–47) is wider than the special problem of polarization by which they are here precisely illustrated. To an intelligence that could follow the play of interaction between the individual molecules of

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* This principle of compensating molecular forcives was briefly enunciated for capillary action and applied by YOUNG in his Memoir "On the Cohesion of Fluids": 'Phil. Trans.,' 1805. It forms the basis of POISSON'S "Nouvelle Théorie de l'Action Capillaire," Paris, 1831, in which the attraction between the molecules of a fluid is balanced by a repulsion of much smaller range, supposed to be due to their caloric: *cf.* especially Ch. VII. *Cf.* also Lord RAYLEIGH "On the Theory of Surface Forces," 'Phil. Mag.,' 1883, 1890, 1892, especially 1892 (1) pp. 209–220: and VAN DER WAALS' "Essay on Continuity of the Liquid and Gaseous States." In these illustrative discussions, in which the intermolecular forces are restricted to a non-polar character, the compensating stress is usually found in the assumption of an intrinsic fluid pressure of range much shorter than that of the attractions between the molecules: the principle however in its general form only asserts that this compensation must exist, and there is no necessity to specify its character.

matter, mechanical forces, in the ordinary sense, would not exist. The actual interactions between the molecules are however necessarily presented to us divided into various statistical groups, which are the subjects of perception by different senses; and it is the business of physical theory to follow out the relations of these different groupings to each other, and to trace them all back into the ultimate unity. The total energy of the molecules of a material body, corresponding to any kind of excitation or polarization, is thus for us made up of various parts. There is a part involving the interaction, with any molecule under consideration, of other molecules at finite distances, which integrates into an energy function of applied mechanical forces of the system, such for example as gravitational or magnetic forces. Of the remainder of the energy, which arises from the mutual actions of neighbouring molecules, a regular or organized part can be separated out which represents the energy of elastic stress, and is a function of the deformation of the element of volume treated as a whole: this stress arising from the immediate surroundings in part compensates, for the element of mass under consideration, the applied mechanical forces aforesaid. The remaining, usually wholly irregular, parts of the local intermolecular forces and motions compensate themselves mutually on the spot,—or at any rate can be considered as thus compensated by other such forces, of different origins, that are not at present under consideration.* The temperature depends in fact on this irregular *residuum* of forces, and so do the density and the other physical properties of the medium, which are thus affected when, owing to polarization or other excitation, this local part of the molecular forces and motions is altered. If we adhere to these principles, it will not be allowable, in deriving the applied bodily forces of a polarized material system from its organized energy of polarization, to vary such physical constants of the element of mass as occur in the expression for the energy; for we should thereby be trenching on that part of the energy whose variation is compensated molecularly without directly originating transmitted bodily forcive, *cf.* § 63.

49. It seems desirable to have names for the two parts into which the total energy of the molecules of a material medium is thus divided. If we agree to maintain the original precise meaning of the term mechanical (as above employed), viz. that a mechanical force is one which we can actually control for doing work for our purposes on matter in bulk, in contrast with a molecular force which we can reason about but not directly employ, we may call the regular part the *mechanical* energy, and the remaining wholly irregular part the *non-mechanical;* we may also use (as above) the

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* The principle of D'ALEMBERT, which is the basis of the dynamics of finite material bodies, necessarily involves this order of ideas. That part of the aggregate forcive on the molecules in the element of volume which is spent in accelerating the motion of that element *as a whole*, is written off; and the regular part of the remainder must mechanically equilibrate. But the wholly irregular parts of the molecular motions and forces are left to take care of themselves; which they are known to do for the simple reason that the constitution of the material body is observed to remain permanent.

terms *organized* energy and *unorganized* energy with the same meaning, the reference being now to the material medium as a continuous organic whole, transmitting applied forces by stress, not as a numerical aggregate of separate molecules. But it is to be observed that the distinction which is thus intended to be made is not the same as the thermodynamic division into *free* and *bound* energy, employed by VON HELMHOLTZ, which is itself precisely equivalent to the earlier division into *available* and *dissipated* energy, formulated by Lord KELVIN and RANKINE. The energy which in its actual condition is as regards direct mechanical effect unorganized, may become in part organized by aid of a physical transformation involving sifting processes of molecular fineness, which are necessarily non-mechanical and have no place in the dynamics of finite bodies. Thus the unorganized energy of two masses of different gases, at the same temperature and pressure, may be in part converted into organized energy and so into mechanical work by allowing them to transpire into each other across a porous partition, the diameters of whose pores approach molecular dimensions; and the transformation in this case shows itself in a resulting fall of temperature, when the work has been done. In the same way mechanical work may be derived from the unorganized energy of liquids by utilizing osmotic pressure; and the stores of energy of chemical combination of electrolytic substances, which as it exists in them is unorganized, can be largely utilized by making use of the sifting agency of electric force on their dual constituents. All these unorganized energies are therefore in part thermodynamically available, and others not now available may become so by means of yet undiscovered processes. But the unavailable or bound energy of thermodynamics is the *residuum* which we cannot render mechanical by any sifting process in bulk, or by anything short of the application of constraint to the individual molecules. This *residuum* may not be absolutely irreducible, but as the knowledge of physical transformations increases, some parts of it may be raised into the domain of available energy: on the other hand the recognition of temperature coefficients in reversible processes will show that some energies previously considered as wholly available are really in part unavailable. Each such discovery in fact involves an amendment or improvement in the corresponding thermodynamic relations; a process which has happened, for example, with respect to Lord KELVIN'S law of electromotive force of a voltaic cell.

50. Once the idea of temperature is acquired, the whole science of Thermodynamics is implicitly involved in the principle of dissipation, that the unavailable part of the energy of an isolated material system always tends to increase, never of its own accord to diminish. The inference follows directly from this principle, by the reasoning first employed by SADI CARNOT, that if the system pass from a state A to a state B such that it can retrace its path back to A, the unavailable part of the energy is not changed: thus there is a whole "plane" or *complexus* of states, with perfect continuity of transformation among them, so that any one state is freely convertible—whether the process has been actually discovered or not—with any other for which

the available energy is the same, by transition through any intermediate series of these states; and we can pass continuously from one such *complexus* to the others in which the whole series of possible states are included, by additions of available energy to the system. The available energy is thus an analytical function of the physical condition of the system, including its temperature; and the trend of spontaneous change in an isolated system is in the direction in which this function diminishes, the positions of stability, as regards mechanical and thermal and also constitutive disturbance, being those for which it is a minimum. The circumstances of all *steady* configurations of matter, whether static or kinetic, are determined by this law.* It is more direct to state the proposition in the form that the unavailable energy tends to a maximum, the presumption being that sensible energy is available until it is shown to be otherwise. This principle, that energy tends to become mechanically disorganized, or that it never spontaneously tends to organize itself, cannot from its nature be other than axiomatic: and the formation of the available energy function for the different states of matter is then the main business of Thermodynamics. The reversible processes which thermodynamic argument employs are ideal types of regular change, theoretically realizable by mechanical constraints which do not control the individual molecules—the limiting forms, it may be, of imperfectly reversible changes which we can actually produce; the states of matter thus derivable from each other are shown, from the equality of their available energies, to have definite mutual relations which are independent of the ideal process (or construction, to use a geometrical analogy) by which the transitions between these states have been imagined.

The really abstruse abstract problem of the subject is that of the nature of temperature; and the principle most in need of elucidation is that, when a body A is in thermal equilibrium with B, and also B with another C, then A would be in thermal equilibrium directly with C. The most definite thermal specification of a body is the quantity of energy it contains; two bodies are in thermal equilibrium when there is no tendency for energy to pass from one to the other, independently of change of molar configuration or molecular constitution; they are then said to be at the same temperature. The *rationale* of this transfer of energy has been made out for the case of gases, where the exchange takes place in encounters between the molecules, so that there is no tendency to transfer from one mass of gas to another in contact

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* In so far as our constitutive knowledge of material systems relates merely to comparison of different steady states, it can be wholly based in WILLARD GIBBS' manner, like ordinary statics, on relations of available energy of a simply additive character: it is where our knowledge becomes more intimate, and we attempt to trace the courses and rates of kinetic phenomena, for instance in material kinetics, electrodynamics, optics and vibratory phenomena in general, that the simple relations of energetics become insufficient as a mathematical basis for general physics. The principle of available energy suffices for tracing the relations of matter in bulk through the various steady phases in which *ex post facto* it is found to exist: but the genesis of these phases is expressly excluded from its domain.

with it if the mean translatory energy of the molecules is the same for each. This principle of temperature-equilibrium shows that, for all states of matter, the equilibrium of energy between bodies in contact in a steady state involves that a definite molecular relation of the one body shall equilibrate a definite molecular relation of the other: and its universality requires that this relation, whatever it may prove to be, shall be a very fundamental one.

51. It would seem that we can make at any rate an advance towards a complete view by realizing that, even if our sensations of heat had not compelled us to assign a fundamental place to temperature in the physical scheme, the principle of negation of perpetual motions must have led to the formulation of that conception, just as it has in fact led to the conception of potentials. If thermal equilibrium between two homogeneous bodies A and B in contact were not conditioned merely by some physical property of A alone being equal to some property of B alone, then if we had A in contact with B, and B with C, each in a state of equilibrium, and, removing B by mechanical means, moved A into direct contact with C but with such ideal constraint applied to the matter in bulk that chemical action is prevented, the physical state of each of these latter bodies would become changed, involving the performance of mechanical work; and a self-acting cycle could be designed by which we might thus obtain an unlimited quantity of work, that is, so long as there remained any diffused molecular energy to be converted. Hence in equilibrium there must be a property, namely the temperature, of each individual body in the field that has the same value for all of them: although of course this does not prevent us from imagining a partition or constraint, nearly adiathermanous, across which such equilibrium would be established as slowly as we please. It follows also that equilibrium of temperature must be the same whether it is brought about by conduction or by radiation. Temperature, as thus introduced, has nothing to do directly with the field of force in which the body is situated: for the relations of bodies to fields of force, in which they are moved about, are treated independently in the consideration of energy relations, and must not be introduced twice over,—or, in other words, the perpetual motion principle can be directly applied.

The single fundamental principle, on which all thermodynamic and thermochemical theory rests, would thus be the axiom of the negation of perpetual motions: and this stands rather in the relation of a principle that could hardly be conceived to be otherwise on any feasible physical scheme, than of one of which we can expect to offer any formal demonstration. Various essays have been made to deduce CARNOT'S principle and a dynamical specification of temperature from special hypotheses as to molecular action: it may be held that, in so far as these are useful it is by way of illustration. It is even possible to conceive, but only in a highly abstract sense, that thermodynamics might have been developed in CARNOT'S manner out of the perpetual motion axiom alone, without the aid of JOULE'S demonstration of the nature and measure of heat; there would then have been merely no knowledge of what had become of energy that had ceased to be mechanically available. It is thus the principle of the limited conservation of available energy, rather than the complete conservation of total energy, that reigns in general non-molecular physics.*

There is still however the complication that the available energy of a system is not a function of its state alone, but involves comparison with some standard state into which it is possible for the system to be transformed. To find the extent of this undetermined element, let us simplify the relations in the ordinary manner, by adopting the scale of temperature that is given by the expansion of an ideal perfect gas, and find out how much energy is dissipated or lost to available mechanical effect, when a quantity of heat H_1 is abstracted at the temperature T_1 and of it H is returned at the temperature T. If all possible mechanical effect were produced, only $H_1 . T/T_1$ would be thus returned instead of H : hence the dissipation is $H - H_1 . T/T_1$ or $T(H/T - H_1/T_1)$. Thus an operation of this kind which does not involve dissipation does not alter H/T : and by accumulation of such changes it follows that any two states of the system which are convertible without dissipation have $\Sigma H/T$ the same for both. The entropy function ϕ of CLAUSIUS thus necessarily enters into the analytical formulation of the principle of mechanical availability. Between a standard state at temperature T_0 and another state at T the dissipation is $T(\phi - \phi_0)$; thus the available energy A in the latter state is $E - T\phi + T\phi_0$, where E is the total energy which involves an undetermined constant part, and ϕ_0 is another undetermined constant which represents the entropy of the system in the standard state. The temperature of the standard state to which the system is referred could not of course be the ideal, practically infinitely remote, temperature which is called absolute zero ; that would imply that the energy is all mechanically available as in ordinary statics.

The presence of this undetermined multiple of T does not really restrict the application of the theorem of minimum availability : it merely implies that when once mechanical and constitutional equilibrium has been determined at any assigned temperature by making A a minimun with respect to the other independent variables, still further degradation will occur if opportunity is allowed for fall of temperature by escape of energy from the system. All that it is necessary to ascertain in any problem is the equilibrium as regards physical state and chemical constitution *at each temperature*, and the capacity of the system for heat, which specifies the thermal change that occurs when the temperature is altered. There is no restriction involved in taking the temperature the same throughout the system, for that is a necessary condition of equilibrium : when it is convenient to imagine partitions impervious to heat, the parts of the system thus separated can be treated as independent systems. The available energy, here arrived at directly from the

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* This seems to be substantially the position which RANKINE took up in 1853 ("Scientific Papers," p. 311) : *cf.* also the weighty introduction to "Outlines of the Science of Energetics," 1858, *loc. cit.*, pp. 209–220. It is in fact the standpoint of CARNOT'S "Reflexions."

perpetual motion postulate, is the same as the free energy of VON HELMHOLTZ'S exposition: he has explained ("Abhandlungen," II., p. 870) how its form can be experimentally ascertained for the different phases of matter, except as regards an undetermined part, as above, of form L + MT, where L and M are constants; that then the equilibrium state of a system of reacting bodies at any assigned temperature is the one that makes it minimum for that temperature, thereby formulating the general solution of the problem of physical and chemical equilibrium: while the other properties of the system, heat-changes and heat-capacities, as well as total energy and entropy, are obtained from it directly by processes of differentiation. The available energy is thus a single characteristic function which includes and determines completely the circumstances, mechanical, thermal, and constitutive, of the steady states of an inanimate material system.

Application to Fluids: LAPLACE'S *Intrinsic Pressure: Law of Osmotic Pressure: Laws of Chemical Equilibrium.*

52. In an incompressible fluid medium in equilibrium, no part of the bodily extraneous forcive is compensated by reaction arising from special strains produced around the element of volume itself; it is all transmitted by fluid pressure independently of the special physical constants of the medium. For equilibrium to subsist in a polarized fluid, the applied mechanical forcive must simply be derived from a potential. When the induced polarization follows a linear law, this potential must also be equal and opposite to the organized energy induced per unit volume in the medium on which this extraneous forcive operates; for the total organized energy that has been spent in the polarization of the element $\delta\tau$ is equal to $\delta\tau$ multiplied by the scalar product of the polarization and the polarizing force, and of this one half is mutual energy of the polarizations of the elements of volume and one half is mechanical work done in the process (*cf.* § 71). If therefore the organized energy of the internal excitation of the medium is expressible as a volume-density of energy represented by a continuous function, the fluid medium will be in internal mechanical equilibrium: but if that function is discontinuous so that in crossing some interface the density of induced energy abruptly changes its value (as for example may be the case when the interface separates two different substances) then in order to maintain equilibrium the applied forcive must include a traction applied to this interface along its normal, of intensity equal to the difference of the densities of energy on its two sides, and acting towards the side of smaller density of energy. At an external boundary there must similarly be applied an outward traction along the normal, equal in intensity to the density of organised energy induced in the part of the substance that is just inside.

To illustrate and elucidate this by the electric phenomena, consider the interface between two dielectric fluids to be maintained in position by an applied traction:

let an element δS of it sustain a displacement δn along the normal, of amount very slight compared with the linear dimensions of δS. If no other boundary within the range of the electric field is thereby affected,—for instance if each fluid is supposed to be continued in a narrow tube to a great distance beyond the field and simply advances or recedes in the end of this tube,—the change of organized electric energy is merely the substitution of a volume $\delta S \delta n$ of energy of the one intensity for the same volume of the intensity on the other side of the interface. The displacement of δS of course affects the state of the field all over, but by hypothesis the electric field was in internal equilibrium, so that the change of the organized energy of any volume-element of the mass arising from a slight derangement is of the second order of small quantities, and produces no sensible effect. The above change of energy is thus equal to the work done by the extraneous traction over δS; which confirms the result already obtained (§ 37) by detailed analysis of the polarization, that the traction is along the normal to the interface and equal in intensity to the difference of the densities of the organised electric energy on its two sides.

53. It is advantageous in connexion with this subject to form a definite conception of the transmission of ordinary mechanical pressure in a liquid. Let us imagine an ideal infinitely thin interface in the fluid: what concerns the equilibrium of the fluid on one side of it is not the pressure which that fluid exerts on the interface, but the forces that are exerted on that fluid itself both by the interface and by the molecular attraction of the fluid on the other side of it. As the range of molecular attraction is very small, these forces together make up a pressure on the fluid, equal in circumstances of molecular equilibrium to the resistance of the interface against the impacts of the molecules diminished by the attraction exerted on these molecules across the interface; and this is the pressure that is transmitted by the fluid. For imagine a canal or tube in the fluid, with infinitely thin sides, and of diameter large compared with the radius of molecular action, and consider the equilibrium of the mass of fluid contained in it between two cross-sections A and B. There will be this pressure acting on the fluid just inside A, and a similar pressure acting on the fluid just inside B; and unless these are equal, or balance each other with the aid of extraneous applied forces such as gravity, the mass of fluid cannot be in equilibrium. This is PASCAL's principle, that the mechanical pressure is transmitted unchanged in amount, except in so far as it is compensated by extraneous mechanical forces. It is to be noticed that the argument does not assume that the fluid between A and B is homogeneous, all that is required is that it be in equilibrium; the cross-sections A and B may be in different fluids, with an interface between, and, provided the diameter of this ideal canal is large compared with the radius of molecular action, the interfacial forces will practically all be mutual ones between molecules inside the tube, and so will not affect the transmission of pressure. It is this transmitted pressure that is the subject of actual measurements: for example in ANDREWS' experiments on the compression of carbonic acid, it is the pressure

so transmitted through the mercury into the companion manometer tube containing perfect gas that is measured and is represented on his diagram of isothermal lines. The two terms of which it is the difference, namely the reaction of the interface against molecular impacts, and the molecular attraction across the interface, are separately represented in VAN DER WAALS' characteristic equation. When the virial equation of CLAUSIUS is applied to a mass of liquid with a free surface abutting on a gaseous atmosphere, there results the relation that the pressure of this atmosphere against an outer boundary, which is the same as the transmitted pressure in the liquid, is equal to two-thirds of the part of the mean density of kinetic energy in the liquid that is connected with encounters and mutual forces between the molecules, together with one-third of the mean virial per unit volume of these intermolecular forces, the latter part being negative and, if polar forces could be assumed absent, of CLAUSIUS' form $\frac{1}{3}\Sigma Rr$; and this without reference to the character of the transition between liquid and gas at the free surface. When on the other hand the virial equation is applied to a mass in the homogeneous interior of the liquid, bounded by an infinitely thin interface, the virial of each molecule vanishes because the attractions acting on it compensate each other on the average, and the result is that the kinetic pressure exerted by the fluid on this interface is simply two-thirds of the mean density of kinetic energy of the bodily motions of the molecules, their internal constitutive energies being excluded.* It follows that the mutual molecular attraction across the interface produces a pressure on the interface from each side equal to the mean virial per unit volume; as in fact would flow directly from the principle that two statically equivalent force-systems have the same virial.

54. Let us construct as above an ideal rigid tube, with infinitely thin walls which exert constraint on the molecules but no attraction, having one of its open ends A in the liquid and the other B outside it; but let us now suppose that the diameter of the tube is small compared with the radius of sensible molecular action, which implies that this radius extends over a considerable number of molecules. The molecular forces acting on each molecule in the tube, whether near the end of it or not, are now almost wholly due to molecules outside it, and are on the average self-balancing, except in the case of molecules at the free surface which are subject to the whole inward molecular attraction of the liquid. The equilibrium of the contents of the tube, which are liquid in one end and gaseous in the other, therefore requires that the kinetic pressure on the molecules in the liquid end A exceeds that on those in the gaseous end B by a constant amount, namely the pressure due to the inward attrac-

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* Some consideration is required as to the omission of the virials of mutual forces acting inside the separate molecules: these must be taken as wholly compensated by kinetic energy of internal motions not thermal, which is legitimate in so far as molecular encounters do not sensibly excite radiation but only slow free precessional motions, and so do not sensibly disturb the configuration of the internal dynamical system of the molecule.

tion exerted on the surface molecules in the layer of transition. It follows that the pressure of molecular attraction across an internal interface, which is the virial per unit volume with changed sign, is equal to LAPLACE's intrinsic pressure K in the liquid arising from the inward attraction of the surface molecules. This equality is easily seen to involve the consequence that the layer of transition at the free surface is very thin compared with the radius of molecular attraction, an important conclusion of which the bases are here the statistical stability of the liquid state, the dynamical principle of the virial, and the hypothesis that the range of sensible molecular attraction extends over a considerable number of molecules in the liquid state. In the condensation of a vapour there is degradation of internal energy into sensible heat of amount equal to the latent heat of condensation diminished by the work of condensation of the vapour and increased by the volume of liquid thereby produced multiplied by the LAPLACIAN pressure K.

55. Consider two fluids, one the pure solvent and the other a solution, separated by a rigid porous partition, with extraneous pressure applied on the side of the solution to balance the osmotic pressure and so to produce equilibrium as regards transpiration through the partition. Now let a slight amount of transpiration occur by very slightly reducing this extraneous pressure: thereby work is done against that pressure, equal to its intensity multiplied by the change of volume owing to transpiration of the solvent into the solution. The operation takes place steadily under conditions of equilibrium, so that it can be reversed either by a known process or, as we might assume, by some process not yet discovered—in this case merely by reversing the pressure, or it may be cyclically by evaporation: thus the work is done at the expense of an equivalent of available energy, partly thermal, and partly of a molecular type which would otherwise run down into heat of mixing of the liquids. Hence the osmotic pressure between two fluids is equal to the whole amount of free or available (not total) energy that would be degraded when unit volume of the pure solvent is mixed with an indefinitely great volume of the solution into which it transpires, supposing that there is no sensible change of volume in that process; if there is change of volume this value must be altered in the ratio of the final to the original volume of the transpired material: so long as the dissolved molecules are out of each others' range of influence, the change of volume, if any, must be independent of concentration. This proposition will be exactly true if the pores in the partition are so narrow, that the cross-sections of the filaments of fluid contained in them each involve so few molecules that the mutual energy of the molecules of fluid in the pores is negligible compared with that of an equal mass of fluid in bulk. Inasmuch as to excite the osmotic pressure, pores or tubes of molecular fineness have to be employed, it follows that it is not an ordinary transmitted mechanical pressure; and the energy which is associated with it is not merely the organized energy from which the mechanical forcive is derived, but the whole amount of energy thermodynamically available. If the pores are wider, the mutual energy of the molecules in them ceases

to be negligible; the effective osmotic pressure then diminishes, being accompanied by diffusion in the pores which involves dissipation of energy that would otherwise produce osmotic effect. Thus the proposition that the osmotic pressure between two fluids is equal to the free or available energy of mixture per unit volume of transpiration, gives only the limiting value which applies to partitions with pores sufficiently narrow. In the equilibrium stage of transpiration through a colloid membrane, operating by absorption into one face of the membrane and evaporation from the other, the limiting pressure may however be reached, provided the action does not involve irreversible thermal processes in the membrane.

The osmotic pressure between a solution and the pure solvent is, from another point of view, the mean aggregate of the forces that have to be applied to the individual molecules of the dissolved substance in order to prevent them from travelling across the interface into the pure solvent, whether that forcive be applied by the resistance of a material partition, or as in the case of ions diffusing across the interface between two salt solutions in contact, by the pull of the electric field which the diffusion has produced—the unmodified molecules of the solvent being in each case free to move either way. Viewed in this light, there is nothing occult or merely analogical—unless it be the presence of ions—in the principles by which NERNST determines the constitution of the layer of transition which gives rise to the potential difference between two salt solutions, and so determines the voltaic and thermoelectric differences of potential at such transitions, by balancing a bodily force arising from osmotic pressure by another arising from the electric field due to the reacting double layer generated by the diffusion.

56. Suppose that the pressures on the two sides of a porous partition separating dielectric fluids are adjusted so that there is no flow across it. When an electric field is introduced this equilibrium is destroyed by the effective electric tractions on the interfaces of separation between the dielectric fluids in the individual pores. To re-establish equilibrium a difference of pressure at the two sides of the interface, equal to that of the electric tractions (§ 37), must be called into play: that is, an electric field influences the value of the osmotic pressure between dielectric fluids. This effect is of course directly connected, through a cyclic process, with an influence on vapour tension (§ 81, *infra*). Its amount is to a great extent independent of the size of the pores; though when the pores are of molecular dimensions it mainly arises from a bodily forcive on the contained filaments of fluid. This electric osmotic pressure will then even hold good with respect to liquids which readily mix; for the obliteration of the sharpness of the interface in the narrow tubes or pores of the partition will take place very slowly, while the formulæ of this memoir for electric tractions are precisely those which hold good for a gradual transition.

This action is different from the one discovered by QUINCKE and discussed by VON HELMHOLTZ,* forming in fact a further extension of the scope of the principle of

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* VON HELMHOLTZ, "Studien über electrische Grenzschichten," 'Wied. Ann.,' 7, 1879.

electrolytic dissociation, in which a stream of conducting fluid forced through a porous non-conducting partition produces an electric current across it, and conversely an electric current forced across the partition carries the fluid with it. Over the surface of each pore there is, on the present view, the intrinsic static potential difference between the partition and the fluid, due to strong orientations of the polar molecules of these two media which lie along the interface, under their mutual influence which stands in place of VON HELMHOLTZ'S attraction of matter for electricity as the exciting cause of voltaic phenomena;* and this difference will be in time diminished by the presence of free ions which become attached among the outward-pointing poles, thus constituting a reverse potential difference with which electro-capillarity deals. The flow of fluid through the pores carries on some of these ions along with it, which thus constitute the observed electric current. Sudden diminution in the extent of the surface would act similarly by crushing them out, as in the observed electrification near waterfalls: rapid extension of the surface, as in the formation of drops in air, should conversely eliminate the effect of the ions bound to the polarized air-film on the surface, by spreading them over a wider area, and so increase the potential difference towards the limiting statical value. On the other hand, when the media in contact are very dilute electrolytic solutions in the *same* solvent, the calculations of NERNST show that the potential difference is wholly an affair of ionic diffusion, as indeed it must be if the efficient polar molecules are all ionized; in that case the normal potential difference will require a sensible time to become established. When, in the case of a mercury electrode dropping into an electrolytic solution, sufficient time is not allowed, the part of the actual potential difference which arises from this cause and not from the intrinsic statical orientation of the molecules, will tend to a vanishing limit, except in so far as it is continually restored by a polarization current in the electrolyte.

57. In the case of very dilute solutions it is possible to obtain a definite expression for the limiting, or maximum, osmotic pressure. After a certain stage of dilution, each dissolved molecule is effectively out of touch with its fellows and is completely environed by a collocation of molecules of the solvent: further dilution therefore does not involve any sensible change in the mutual free energy of the solvent and the dissolved molecules; all that occurs is a wider separation of the dissolved molecules in space, with such energy changes as may be directly concerned in it. Suppose now that the dissolved substance is a gas, and that the solution is separated from the pure solvent by a partition which the latter can traverse while the gas cannot: whether such partitions are known to exist is inessential to the theoretical argument, the function of the partition being merely passive constraint exerted on the *aggregate* of

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* [HELMHOLTZ had to be content in his analysis with the crude conception that different kinds of matter attract electricity differently. On a scheme like the present the obvious explanation is that the polar molecules of the two substances act on each other across the interface, producing a certain regularity of orientation which forms the intrinsic double layer to which the potential difference is due.]

the dissolved molecules. The solvent will transpire across the partition into the solution, unless a definite osmotic pressure acts against it, when there will be equilibrium. Let us examine the change of free energy involved in the very slow transpiration of a certain volume; all essential that has happened has been an expansion of the molecules of the contained gas, each with its fluid environment, into a larger space. We may compare the two states of the gas, as it would exist free with these two different volumes, and then suppose that by an ideal process the fluid environment of the molecules is directly brought about in each case: that process will, as regards change of intimate molecular configuration, be essentially the same for both states of the gas, therefore the change of free energy due to the dilution of the solution is simply that which corresponds to the free gaseous expansion of the dissolved gas.* This conclusion carries with it, by the thermodynamic principle of free or available energy, a theoretical proof of VAN'T HOFF's generalization that the osmotic pressure of a very dilute solution is equal to the gaseous pressure of the dissolved molecules when they are supposed to occupy the same volume in the gaseous state. The extension of this proof to dissolved liquids and solids, which form the practically important case, is at first sight barred (unless it is formulated as in the footnote) by the fact that we cannot then actually have the molecules existing free at the same volume as they occupy in the dilute solution. But when the ANDREWS isothermal for the dissolved substance is made into a continuous curve by inserting a supersaturated wavy part, there will always be a real point on it corresponding to the volume occupied by the substance thus existing in a homogeneous condition, and also a corresponding pressure which at the small density under consideration would practically be that of the gaseous state: thus there would be no difficulty in the extension to dissolved solids and liquids, were it not that this point on the isothermal might be on the thoroughly unstable reach, along which rise of density corresponds to fall of pressure, so that any slight accidental inequality of density would be spontaneously increased. The successful use made of the ANDREWS diagram for numerical calculation of the properties of substances by VAN DER WAALS shows however that its physical reality is not destroyed by this instability: and when it is remembered that instability can be theoretically removed by slight constraint which does not sensibly affect the material transformations and does not affect the energy relations at all, it will appear that there is good reason for generalizing the law of osmotic pressure above demonstrated for gases. As before stated, what is most desirable to

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* The circumstance which makes this purely imaginary process legitimate is that the available energy is a function of the constitution of the matter *in bulk*, not depending on the accidental characteristics of state or motion of the individual molecules: now the only change that has occurred as regards the constitution of the substance in bulk, that can affect either the available or the total energy, is the change of volume of the solution by transpiration of the pure solvent across the partition, which by the above affects it in a manner absolutely independent of the nature of the homogeneous solvent, and therefore of the existence of the solvent at all, because the relation of each molecule to the portion of the solvent within its sphere of influence is not changed.

supplement on mechanical principles an explanation like the present one is not so much any accession of logical rigour on ordinary thermodynamic premisses, as some precise notion of what is involved, as regards detailed molecular dynamics, in equality of temperature. In the present differential procedure no assumption has been made on that head; and no inference that osmotic pressure is, like gaseous pressure, due to simple molecular bombardment is warranted. When a theoretical basis is thus found for VAN'T HOFF'S principle, the laws of the molecular influence of dissolved substances on the freezing point and vapour tension of very dilute solutions of course go along with it.

58. It may be objected that the application of the principle to ionized solutions would compel us to admit the possible theoretical existence of a gas consisting of ions: but that is not really so, because the argument only compares one state of dilution with another. Yet on the other hand there is the hypothesis, supported by BRÜHL'S work on optical equivalents, that under certain circumstances oxygen is a tetrad element, so that the molecule H_2O can take up sufficient ions to form another saturated molecule of type $H_2 = O = X$, and that therein lies the cause of the regular ionization current produced by solution in water (the ions X being free only when passing from one such combination to another), as contrasted with the irregular ionization of free gases. Changes of valency in an element remain unexplained, but their occurrence is now usually accepted as matter of fact.* The function of the osmotic diaphragm is merely passive, to prevent mixture by diffusion and consequent loss of mechanical availability. In a mutual solution of two substances, it is usually only the one that is present in large excess that gets through the diaphragm in purity: if it should prove to be a general law that the dialyzing action is only complete when there is such large excess, it would be strong evidence for the view that the molecules dissolved in it form the nuclei of loose molecular complexes which are too large and permanent to get through, while the free solvent in which the molecules are not thus grouped is not so hindered.

When a solution is made more and more dilute, there comes a stage when it would

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* The connexion between the various phenomena may be pictured in neutral terms, as POYNTING has recently done ('Phil. Mag.,' Oct. 1896), starting from a hypothesis that pressure increases the "molecular mobility" of a fluid according to an *assumed* law equivalent to VAN'T HOFF'S principle. In order to evade the hypothesis of partial dissociation in salt solutions, he restricts the sphere of action of a dissolved undissociated molecule to one or two or three definite molecules of the solvent, leading to correspondingly different amounts of osmotic pressure; thus a temporary chemical combination is dealt with instead of, or it may be in addition to, an extended sphere of influence. But the considerations given above show that it is the number of spheres of influence that is really effective, so that if there is chemical combination it must be in part with dissociated ions as in BRÜHL'S view. On the special assumptions involved in the extension of the methods of gas-theory to liquids, BOLTZMANN ('Zeits. für Phys. Chemie' vi, p. 478) has offered a demonstration (approved by LORENTZ) of the law of osmotic pressure, which seems to refer it to molecular bombardment, and require that the mean energy of translation of a molecule shall be the same in the liquid state as in the gaseous state at the same temperature.

seem impossible to imagine that its electrolysis, if it remains of normal type, is conducted through a mechanism like GROTTHUS' chains: the dissolved molecules are far out of each others' range of influence, and the very first stage of the working of a GROTTHUS' chain containing molecules of the solvent would produce that dissociation which it is the object of the chain theory to evade. Similar considerations apply to the velocity of chemical reactions. When a solution of K.HO neutralizes one of HCl, the heat generated is mainly that of the union of H and HO to form H_2O: when the solutions are very dilute this should take a considerable time to develope, even allowing for intimate mixture by stirring, if each H had to find its HO partner directly. The immediate reaction must therefore be due to a mobile equilibrium of dissociation being disturbed by the mixing of the solutions, and then re-establishing itself.* Thus in the progress of an ion H through the water under the electric force in electrolysis, it would not be the same H that is driven on, but that ion often gets fixed liberating another one in its place, so that it is the mean translation of a condition of matters in which there is a definite number of H ions in the element of volume that is given by KOHLRAUSCH'S law, not that of an individual ion. This accords with WHETHAM'S interpretation of his result, that in acetic acid solutions, in which the conductivity is abnormally low, the ionic velocity is abnormal to an equal extent.†

59. A principle quite analogous to the one on which VAN'T HOFF'S law has here been based, has already been applied to a cognate phenomenon in authoritative investigations. The transpiration of two different gases into each other across a porous partition establishes a difference of pressure; there is thus present a store of available energy, which would be run down in the mixing of the gases; its amount, as originally determined by Lord RAYLEIGH from the special properties of gases, is obtained by finding how much free energy runs down when the gases are each separately expanded to the volume of the mixture, and adding these amounts. This result, either in the present form or expressed with reference to entropy, has been sanctioned, explicitly or tacitly, as axiomatic by MAXWELL‡ and other authorities, when applied to gases whose molecules do not exhibit sensible mutual attraction: the change of configuration arising from the two mutually independent systems occupying the same space, instead of different equal spaces, is rightly held to involve no change in the available energy. The principle above employed is of precisely similar nature.

If we imagined two gases in which the molecular mass differed only infinitesimally,

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* In the same way, if a gaseous reaction were of ternary type, so that three atoms or ions had to unite to form a molecule, it must proceed far more slowly than a binary reaction, and may not get established at all, except by the help of the catalytic action of some other substance, such as water vapour, in reducing it to binary stages or facilitating the simultaneous presence of the three kinds of atoms in the same molecular sphere of action.

† W. C. D. WHETHAM, "Solution and Electrolysis," 1895, pp. 142, 155.

‡ 'Encyc. Brit.,' Art. "Diffusion": Collected Papers, II., p. 644.

just the same amount of work could still be gained by mixing given volumes of them in a reversible manner as if they were gases wholly unlike; but the transpiration pressure would then be infinitesimally small and the time of transpiration infinitely great.* It is thus impracticable to proceed to a limit, and no paradox is here involved such as the assertion that a finite amount of work could be gained by mixing two gases which are practically identical in properties. A similar apparently paradoxical limiting case might be formulated as regards osmotic pressure of a dissolved substance very nearly identical with the solvent.

60. The law of HENRY that the density of dissolved gas is in a constant ratio s to its density as it exists free in the surrounding atmosphere, is involved in the osmotic law, and conversely may be employed to verify it. In circumstances of equilibrium the potential of free energy of the dissolved gas (in GIBBS' sense) must be the same in the liquid and the atmosphere; that is, the removal of an elementary portion of the gas from the liquid to the atmosphere must not alter the free energy of the whole. Thus the difference of the free energies of the dissolved gas per unit mass, when its partial pressure in the liquid is changed from p_1 to p_2, is equal to the difference of the free energies of the gas per unit mass in the surrounding atmosphere when its partial pressure is changed from p_1/s to p_2/s. The latter difference is by Lord RAYLEIGH'S principle, independent of what other gases may also be present in the atmosphere: it is thus $\int pdv$, where $pv = R'\theta$ for the unit mass of gas, and is therefore at constant temperature $R'\theta \log p_1/p_2$. This does not involve s, and therefore the *difference* of the free energies of the dissolved gas at two different densities is the same as if it existed in the free gaseous state at those densities: and this carries with it identity for the two states as regards all relations of available energy and work. Conversely, the law of HENRY follows as above, by the principle of available energy, from the circumstance that the molecules of the dissolved gas are outside each others' spheres of molecular action, independently of any picture that we may form of the process of exchanges in evaporation and absorption.

It is a confirmation of the soundness of this thermodynamic theory, that the law of osmotic pressure for dissolved gases is immediately involved in, and might have been predicted from, the equations given by VON HELMHOLTZ in 1883,† in a discussion of their energy relations in connexion with the theory of galvanic polarization.

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* The assumption is involved that the gases are really different and that means exist for separating them. [The fact that the amount of available energy at our command depends on the control we have learned to exercise over physical processes does not detract from the objective validity of that conception as a deduction from general principles of molecular theory, as has often been suggested (*cf.* §49 *supra*): any more than our possible complete ignorance of some forms of total energy would give to the idea of energy itself a subjective aspect.]

† "Zur Thermodynamik Chemischer Vorgänge," III., 'Monats. Berl. Akad.,' May, 1883, especially equations (4) and (5); 'Abhandlungen,' III., pp. 101–114. The law had however been arrived at quite explicitly by WILLARD GIBBS as early as 1876 in his discussion of the general theory (p. 226). Recently the argument has been carefully formulated by Lord RAYLEIGH, 'Nature,' Jan. 14, 1897: *cf.* also a

61. It is the circumstance that the available energy A of § 51 is a function of the bodily configuration and constitution of the system, whose alteration by dilution is independent of the nature of the solvent provided the solution is sufficiently dilute, that makes osmotic pressure independent of the solvent and therefore the same as the corresponding gas-pressure. This is of course different from asserting that the whole available energy of a dissolved substance is the same as its available energy at the same density in the free gaseous state. In fact the difference between these energies may be estimated from a knowledge of the solubility: thus the available energy per unit mass of the gas in the solution at its actual density ρ' is equal to that of the same gas in the free space at the corresponding density ρ; so that the available energy per molecule of the dissolved gas is equal to that of free gas of its own density and temperature together with $R_1T \log \rho'/\rho$ and also R_1T for the volume occupied by the free gas. This makes in all for the excess of available energy, per molecule, of the dissolved gas $R_1T \log e\rho'/\rho$, or $R_1T \log es$, where s is the solubility and R_1 is a gas constant the same for all kinds of molecules. Like information is derivable from the ratio of partition of any dissolved substance between any two solvents which do not intermix: its available energy per unit mass must in the state of equilibrium be the same in both solutions.

62. The increase of available energy involved in molecules or atoms of given species appearing in the dilute solution during chemical change is, per molecule, $a - R_1T \log bN$ where N is the number of such molecules already there per unit volume and a is a function of the temperature, b being a constant which depends on the standard temperature of reference (§ 51). A reaction going on in the solution involves the disappearing by breaking up of molecules of some of the types present, and the appearing of molecules of other types to an equivalent extent: when chemical equilibrium is attained, the change of available energy arising from a slight further transformation of this kind must vanish: that is,

$$n_1(a_1 + R_1T \log b_1N_1) + n_2(a_2 + R_1T \log b_2N_2) + \dots .$$

vanishes, leading to $R_1T \log b_1^{n_1}b_2^{n_2} \dots \log N_1^{n_1}N_2^{n_2} \dots = -(n_1a_1 + n_2a_2 + \dots)$, where $n_1, n_2, \dots$ are the numbers of the molecules of the different types that take part in the reaction, reckoned positive when they appear, negative when they disappear; so that $N_1^{n_1}N_2^{n_2} \dots .$ is equal to K, a function of the temperature, which is the law of chemical equilibrium originally derived by GULDBERG and WAAGE from statistical considerations. Again, if A is the available energy of the whole solution, and δA, equal to $\delta A_0 + R_1T \log K'$, where $K' = Kb_1^{n_1}b_2^{n_2} \dots .$, denotes its variation per

letter by GIBBS, March 18. The pressure difference is necessitated by the circumstance that the steady state would be brought about by interchange of individual molecules. But its amount is calculable *a priori* only when the dissolved molecules are practically out of each others' range: and then the argument in the text shows that it depends solely on the number of molecular aggregates with foreign nuclei that are present, irrespective of whether these nuclei are complete molecules or parts of dissociated molecules.

molecule of reaction without change of temperature, δA is null as above in the equilibrium state at each temperature, so that $\delta A_0 = - R_1 T \log K'$. And, with partial differentiation, $d/dT\ (\delta A/T) = d/dT\ (\delta A_0/T)$, hence it is equal to $- R_1\, d/dT\,.\log K'$ or $- R_1\, d/dT \log K$, and is independent of the unknown term A_0. Now reverting to the general theory, if E is the energy in a system, dH the heat imparted to it and dW the work done to it, $dE = dW + dH = dW + T\, d\phi$ and $A = E - T\phi + T\phi_0$, where A, E, T, ϕ are all analytical functions of the state of the system. Thus, employing total differentials, $d(A/T) = -E\, dT/T^2 + dE/T - d\phi = -E\, dT/T^2 + dW/T$: so that in the present case $d(\delta A/T) = - \delta E . dT/T^2 + d\, \delta W/T$ If the small amount of reaction represented by δ occurs so that no mechanical work is done on the system, δW is null; hence δE is equal to δH the amount of heat taken into the system from its surroundings per molecule of the reaction when it proceeds without work. Thus finally $\delta H = - T^2\, d/dT\ (\delta A/T) = R_1 T^2\, d/dT\ (\log K)$, which is the thermal relation developed by VAN 'T HOFF.*

On the Electromotive Forces established by Finite Diffusion.

63. The function of an osmotic partition in preventing by pure constraint the diffusive degradation of energy (§ 54) is illustrated by the theory of electromotive forces of diffusion. In the concentration-cells, of which the theory was established by VON HELMHOLTZ, the solution in each cell was homogeneous, and the influence of concentration was determined by balancing different cells against each other: there being no diffusion, the process was reversible, and thermodynamic formulæ were applicable. By forming an electrode of a metal surrounded by one of its insoluble salts, such as mercury surrounded by calomel, employing for the other one zinc immersed in zinc chloride solution, the net constitutive change at the mercury electrode when electricity passes through the cell is independent of the concentration of the solution, being simply the deposition of the equivalent quantity of mercury from undissolved calomel: hence that electrode accounts for a constant part of the electromotive force. On the other hand the change of free energy by dissolution of the equivalent of zinc is made up of a part arising from change of chemical constitution and another part depending on the concentration of the solution which receives the resulting chloride. The part of the electromotive force depending on the processes at the zinc electrode is thus in the case of a reversible electrode equal to const.—$RT \log p$, where p is the osmotic pressure of the zinc chloride solution and R

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* *Cf.* WILLARD GIBBS, *loc. cit.* p. 231, where the case of gaseous reactions was treated. More directly, we can form a reversible CARNOT cycle in which the constitutive change δA is made at temperature T and unmade at $T - dT$. The work of the cycle must be $\delta h\, .\, dT/T$, where δh is the heat absorbed in the change when the maximum amount of mechanical work is done in it by osmotic or other appliances: thus $\delta h\, .\, dT/T = dT\, .\, d\, \delta A/dT$, so that $\delta h = T\, d\, \delta A/dT$. When no work is done in the change, the heat absorbed is δH, equal to $\delta h - \delta A$, which is $- T^2 d/dT\ (\delta A/T)$ as above.

is now the gas-constant belonging to an electrical equivalent per unit volume, which is 8580 in c.g.s. units: this may be expressed in the form RT log P/p, where P depends on the metal of the electrode and on the solvent employed in the cell and on the temperature, but not on the concentration of the solution. This quantity P has been called by NERNST, on grounds of analogy, the solution pressure of the metal electrode in the solvent.* When the electrode is polarizable so that the processes are not reversible, the difference of potential must be less than this formula would give. If now we are dealing with a two-fluid cell, in which the fluids are separated by an osmotic partition and passage of the solvent is prevented by balancing the osmotic tendency by hydrostatic pressure, the processes are still reversible and the electromotive force of the cell will be RT ($\log P_1/p_1 - \log P_2/p_2$), while passage of a current will gradually polarize the faces of the partition. If however the ions could pass through the partition into the solution of different concentration without diffusion of the fluids in bulk, the part of this electromotive force depending on concentration would be cancelled, and there would remain RT log P_1/P_2 due solely to the affinity of the solvent for the materials of the electrodes. But if we are dealing with a cell, in which the fluids are in direct contact along an interface of finite dimensions so that steady diffusion at a finite rate is going on, or in which they are even allowed to diffuse steadily across an osmotic partition, there will be loss of availability owing to that diffusion, so that the back electromotive force arising at the junction of the fluids is less than the maximum value $-$ RT log p_1/p_2. In the absence of knowledge of the rate at which the diffusive degradation of energy is proceeding and is affected by electric transfer, the principle of availability cannot supply a formula for this diminution of the back electromotive force, which will depend on the nature of the layer of transition: but a theory of the process of steady interdiffusion of two ionized fluids has been formulated by NERNST and PLANCK which involves an expression for its magnitude.† Thus, considering diffusion of a simple

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* There appears a difficulty in imagining, in accordance with the view here taken, that the value of P can be dependent on a layer of the metallic ions extending into the solution, especially as the potential difference between dielectrics could not be so explained. *Cf.* § 56 *supra.*

† I find that applications similar to the above, but on a more extensive scale and with considerable differences in the argument, especially a more prominent use of entropy, are made in PLANCK'S later important exposition "Ueber das Princip der Vermehrung der Entropie," 'Wied. Ann.,' 44, 1891, pp. 385–428. The general formula for the potential difference between two diffusing solutions is there obtained from the variation of an analytical function, which is really the available energy, on the hypothesis that the solutions are in a permanent state of diffusion, determined by NERNST'S principles, in which the concentration varies from point to point so slowly that the diffusive dissipation other than electric may be neglected. *Cf.* also on the history of the subject NEGBAUR, 'Wied. Ann.,' 44, p. 737. In the text above the statements are confined to the case of binary electrolytes.

The development of the laws of chemical equilibrium given in § 60 has also been largely anticipated as to form by PLANCK, 'Wied. Ann.,' 32, 1887: his postulates are however different from those that enter here, where the analysis occurs as an outcome of a general view of the relations of molecules in bulk to the æther and to each other, §§ 11–12. [*Cf.* PLANCK, "Vorlesungen über Thermodynamik," 1897.]

solution across a layer in which the concentration varies, when the steady state is attained both ions must diffuse together at the same rate notwithstanding their different mobilities u and v, measured by KOHLRAUSCH as the values of their mean velocities due to unit electric force. Now the mean steady velocity of migration of a single ion is equal to this mobility divided by its electric charge e and multiplied by the force which causes its motion: this force consists of an electric part $-e d\mathrm{V}/dx$, where V is the electric potential set up during the transition to the steady state of diffusion, and of an osmotic part. To determine the latter, observe that when a solution is separated from the pure solvent by a permeable osmotic partition, the solvent is restrained from passing across only by an osmotic pressure acting against it: this means that to maintain the steady state without diffusion the osmotic partition must exert more pressure by the amount p on the solution than on the pure solvent. If we consider a layer of the actual solution, of cross-section unity and thickness δx, there would thus have to be a bodily force $dp/dx \,.\, \delta x$ exerted on it if the diffusion of its ions were prevented: therefore $-dp/dx \,.\, \delta x$ is the aggregate of the forces acting on the contained ions and producing diffusion, that arise from the gradient of concentration. If n be the number of ions per unit volume, the mean force per ion is thus $-n^{-1}dp/dx$: this is not a mere hypothesis founded on a vague analogy of osmotic pressure with ordinary hydrostatic pressure, but gives a precise measure of an actual force on a constituent of the medium. The number $d\mathrm{N}/dt$ of single ions of either kind that is driven across unit area of a geometrical interface in a solution of varying concentration by these forces is thus given, after NERNST, by

$$\frac{d\mathrm{N}}{dt} = -nu\frac{d\mathrm{V}}{dx} - \frac{n}{e}\frac{u}{n}\frac{dp}{dx}, \quad \text{also} \quad = nv\frac{d\mathrm{V}}{dx} - \frac{n}{e}\frac{v}{n}\frac{dp}{dx}.$$

If D denote the coefficient of diffusion of the solution, $d\mathrm{N}/dt = -\mathrm{D}\, dn/dx$; and by the gaseous law which applies to osmotic pressure of very dilute solutions $p = ne\mathrm{RT}$. Hence immediately

$$\mathrm{D} = \frac{2uv}{u+v}\mathrm{RT}, \qquad \frac{d\mathrm{V}}{dx} = \frac{v-u}{v+u}\frac{1}{ne}\frac{dp}{dx};$$

so that the integrated potential difference $\mathrm{V}_2 - \mathrm{V}_1$ across the diffusion layer is $\mathrm{RT}\,(v-u)/(v+u) \,.\, \log p_2/p_1$. It follows that when steady diffusion is allowed to go on, the back electromotive force at the junction of the fluids is thereby reduced in the ratio of the difference to the sum of the ionic mobilities. The agreement with experiment of these expressions for $\mathrm{V}_2 - \mathrm{V}_1$, and for the ordinary diffusion coefficient D of a solution as thus determined electrically, constitutes two distinct tests of the general validity of this diffusion scheme, and of the hypothesis of independent mobility of the ions of which it is a corollary.

In the case of a solution only partially dissociated, like that of acetic acid referred to in § 58, provided the time of association of two paired ions is on the average large compared with the time of relaxation of the system, these expressions for D and

$V_2 - V_1$ will still hold for the dissociated portion, if u and v denote the actual velocities of the ions when free, not the abnormally small effective velocities as determined by WHETHAM. Thus the total diffusion would now consist of this part belonging to the ionized portion, with coefficient independent of the degree of ionization, together with the actual diffusion of the non-ionized portion. On the same hypothesis the potential difference between the fluids would depend, as might have been foreseen, only on the actual concentrations of the ions in the two solutions, the amount of non-ionized substance being immaterial except in so far as it gives rise to an ordinary contact difference (§ 56): but it may not be computed from the abnormal ionic velocities by the ordinary formula unless the degree of ionization is independent of the concentration.

Critique of VON HELMHOLTZ'S *Theory of Electric Stresses: Electrostriction not due to Mechanical Forcive.*

64. A theory of electrostatic stress in dielectric media, based on the method of energy, and avoiding molecular theory, has been originated by KORTEWEG,[*] formulated in general terms by VON HELMHOLTZ,[†] and further developed by LORBERG, KIRCHHOFF,[‡] HERTZ[§] and others: it is desirable to examine the relation in which it stands to the views here set forth. The investigation of VON HELMHOLTZ postulates a dielectric medium which is effectively continuous, not molecular; also a potential function, that namely of the distribution of uncompensated polarity which represents the electric state of the medium, satisfying a characteristic equation, that of the FARADAY-MAXWELL theory. The energy per unit volume is expressed in terms of this potential, in such form that the variation of the integral which represents the energy for the whole volume leads, *on integration by parts*, to this characteristic equation as one of the conditions of internal equilibrium; the integral is then asserted to be in the normal form, which would mean, in our order of ideas, that it represents the actual distribution of the energy in the medium as well as its total amount. Its variation with sign changed, owing to change of material configuration, should then give the extraneous forcive that must be applied in order to maintain mechanical equilibrium; the variation with respect to the electrical configuration being null, so that electric internal equilibrium is provided for, by the characteristic equation already satisfied. The variation without change of sign should thus give the mechanical forcive of electric origin that acts on the medium. But the *data* do not even on these assumptions suffice to lead to a definite stress-system for the material; a certain geometrical stress-system is merely assumed which yields on the element of

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[*] D. J. KORTEWEG, 'Wied. Ann.,' 9, 1880.

[†] H. VON HELMHOLTZ, 'Wied. Ann.,' 13, 1882: 'Abhandlungen,' I., p. 798.

[‡] G. KIRCHHOFF, 'Wied. Ann.,' 24, 25, 1885: 'Abhandlungen,' 'Nachtrag,' p. 91.

[§] H. HERTZ, 'Wied. Ann.,' 41, 1890: "Papers on Electric Waves," English edition, pp. 259–268.

volume a mechanical forcive the same as the one thus deduced from the energy function. An infinite number of such stress-systems might in fact be specified, for there are six components of stress which need satisfy only three conditions. If however the stress system is required to be symmetrical with respect to the lines of polarization, there is in this respect no indefiniteness (§ 40); and the one given by VON HELMHOLTZ is of this kind. Thus the definite result really deduced by VON HELMHOLTZ from his energy hypothesis is an expression for the bodily mechanical forcive in the polarized medium, the (X, Y, Z) of equation (4) of his memoir; while correlative formulæ are applied by him and by KIRCHHOFF for the bodily forcive in a magnetized medium. These expressions, however, definitely contradict the formulæ of MAXWELL and of the other previous writers for the bodily mechanical forcive in a magnetized medium, which are in general agreement with those developed in this paper: in fact VON HELMHOLTZ translates his formulæ into MAXWELL'S electric stress system, while MAXWELL himself had to invent for the case of magnetic polarization, which was the one he considered, a different stress-system, namely his magnetic stress. As recent writers have in the main tacitly accepted VON HELMHOLTZ'S procedure, it is incumbent on us to assign the origin of this discrepancy; and for this purpose a summary of his method is given, the effect of alteration of the coefficient of polarization arising from strain in the material being for the present left out of account.

65. The organized electric energy in the polarized medium being assumed, from other considerations, to be

$$W = \int K/8\pi \,.\, (dV^2/dx^2 + dV^2/dy^2 + dV^2/dz^2)\, d\tau,$$

where

$$d/dx\,(K\,dV/dx) + d/dy\,(K\,dV/dy) + d/dz\,(K\,dV/dz) + 4\pi\rho = 0,$$

in which ρ is a constant associated with the element of dielectric matter, called the density of its free electric charge, the forces acting will be derived from the variation of W; variation with respect to V leads to the electric forces, and that with respect to the material configuration leads to the mechanical ones. The problem is to determine the mechanical forces when there is electric equilibrium, that is when variation with respect to V yields a null result. The form of W above expressed does not lead to this null result; we can however by integration by parts derive the form $W = \frac{1}{2}\int V\rho\, d\tau$, the essence of this transformation being that in the new integral the distribution of the energy among the elements of volume $d\tau$ of the medium has been altered. This form does not satisfy the above requirement either, but by combining the two forms we obtain

$$W = \int \{\rho V - K/8\pi \,.\, (dV^2/dx^2 + dV^2/dy^2 + dV^2/dz^2)\}\, d\tau,$$

whose variation with respect to V is null as required: although as integration by parts is employed, the variation is *not* null for each single element of mass. This integral is then taken to represent the actual distribution of the organized energy in the medium when in electric equilibrium, and not merely its total amount: and variation of it with respect to the material configuration should on that hypothesis give the actual bodily distribution of mechanical forcive, not merely its statical resultant on the hypothesis that the system is absolutely rigid. Now in finding the variation of W arising from a virtual displacement $(\delta x, \delta y, \delta z)$ of the polarized material, we have to respect the conditions that the free charge $\rho\delta\tau$ is merely displaced, so that by the equation of continuity $\delta\rho + d(\rho\delta x)/dx + d(\rho\delta y)/dy + d(\rho\delta z)/dz = 0$, and also that each element of the material is moved on with its own K, so that $\delta\mathrm{K} + d\mathrm{K}/dx \,.\, \delta x + d\mathrm{K}/dy \,.\, \delta y + d\mathrm{K}/dz \,.\, \delta z = 0$; while things have been arranged so that a variation of V produces no result,—but only however no aggregate result on integration by parts. Unless the transitions at interfaces are supposed to be gradual, and the integration then to extend throughout all space, there will also be direct surface terms in the variation, because the virtual shift of the material leaves a space unoccupied on one side and occupies a new space on the other; thus finally by the ordinary process of integration by parts we obtain for any region

$$\delta\mathrm{W} = \int\left[\left\{\rho\frac{d\mathrm{V}}{dx} + \frac{1}{8\pi}\frac{d\mathrm{K}}{dx}\left(\frac{d\mathrm{V}^2}{dx^2} + \frac{d\mathrm{V}^2}{dy^2} + \frac{d\mathrm{V}^2}{dz^2}\right)\right\}\delta x + \ldots\ldots + \ldots\ldots\right]\delta\tau$$

$$- \int\frac{\mathrm{K}}{8\pi}\left(\frac{d\mathrm{V}^2}{dx^2} + \frac{d\mathrm{V}^2}{dy^2} + \frac{d\mathrm{V}^2}{dz^2}\right)\delta n\, d\mathrm{S}.$$

The coefficient of δx with sign changed has been taken to be the component of the bodily mechanical forcive. But to obtain the total mechanical forcive acting on an element we must retain all the terms in the variation that belong to it, so that it is illegitimate in this connexion to transmit a traction from it to the boundary of the medium by the process of integration by parts. If then we consider the single element of volume by itself, so that in the formula $\delta\mathrm{S}$ is an element of its surface, the forcive on it will be VON HELMHOLTZ's one together with a hydrostatic pressure $- \mathrm{K}/8\pi \,.\, (d\mathrm{V}^2/dx^2 + d\mathrm{V}^2/dy^2 + d\mathrm{V}^2/dz^2)$ acting over its surface; and this complete specification would agree with our previous results, *except* that we have $\mathrm{K} - 1$ in place of K for reasons already assigned. But it would seem that the method thus described must be radically unsound; it would be valid if there were only one medium under consideration, of which W is the energy function: but there is here, in the same space, the æther with its stress and the polarized matter with its reacting mechanical forces, and (§ 6) there is no means of disentangling from a single energy function in this way the portions of energy which are associated with these different effects.

66. There are also subsidiary terms in VON HELMHOLTZ's formulæ, involving the rate of alteration of the inductive capacity of the fluid dielectric by compression, terms which are extended in the work of KORTEWEG, LORBERG, and KIRCHHOFF to include the alterations of the inductive capacity of a solid dielectric produced by the

various types of strain that it can sustain. Their *rationale* is best seen by the more elementary procedure of KORTEWEG, who first introduced them. He considered the following cycle; (i) move up a piece of the dielectric material from an infinite distance into an electric field, (ii) strain it and so alter its inductive capacity and therefore the electric energy, (iii) move it back to an infinite distance in the strained state, (iv) restore it to its original condition by removing the strain. In order to evade perpetual motions, the mechanical work done by electric attractions as it approaches must exceed the work absorbed as it recedes, by the loss of available electrical energy due to strain; and this leads KORTEWEG to terms in the mechanical forcive which depend on the rate of variation of inductive capacity with strain. The process is analytically developed for fluids by VON HELMHOLTZ, by adding on to δK, the variation of K, a part arising from the compression of the material which the virtual displacement involves, namely by adding $-dK/d \log s \,.\, (d\,\delta x/dx + d\,\delta y/dy + d\,\delta z/dz)$, where s denotes the material density: and KIRCHHOFF formulates it for isotropic solid media, replacing $dK/d\log s$ by KORTEWEG's two coefficients which express the actual rates of change of K due to elongations along the line of polarization and at right angles to it. But here again a process of integration by parts comes in, which removes part of the bodily forcive from the element of volume at which it is directly applied to the boundary, and so vitiates the result regarded as a specification of the forcive which produces the actual mechanical strain in the material.

67. Moreover, phenomena of this latter kind are more appropriately investigated as intrinsic changes of the equilibrium configuration of the material arising from molecular actions produced by the polarization, the forcive of the above argument being simply what would be originated if these changes were prevented by constraint. Such deformations of the elements of volume of the material, the result of electrostriction or magnetostriction, may not fit in with each other, and the strain thence arising will originate secondary mechanical stresses: but it appears preferable to keep these distinct from the regular stress which is the effect of the *direct* electric or magnetic action of different finite portions of the material on each other.

This separate procedure may be illustrated by an investigation of the change of intrinsic length of a bar of magnetic material, caused by its introduction into a magnetic field. Clamp the bar to its natural length when at a great distance; then introduce it into the magnetic field so as to lie along the lines of force; then unclamp in such way that it may do as much work as possible in pushing away resistances to its magnetic elongation; finally remove the unclamped bar again to a great distance. If this cycle is performed at a uniform temperature, it follows from CARNOT's principle that there can be no resultant work done in it. Now the work done by the magnetic forces in introducing the bar is $\int I\, dH$, that is $\int (\kappa + Q\, d\kappa/dQ + I\, d\kappa/dI)\, H\, dH$, per unit volume, where κ is the magnetic susceptibility which is presumably a function of the internal longitudinal pressure Q in the bar and of its intensity of magnetization I. The work done in unclamping is $\frac{1}{2}Q_1 l_1$ per unit volume, where l_1 is the intrinsic magnetic elongation and Q_1 the pressure corresponding to the strength H_1 of the part of the field in which it is unclamped. This is on the supposition that the bar is long, so that there are no free magnetic poles near together which would diminish Q by their mutual attraction. The work done per unit volume by the magnetic forces during the removal of the bar is $-\int(\kappa + I\,d\kappa/dI)\,H\,dH$. The resultant work in the cycle being null, we have $d\kappa/dQ\,.\int QH\,dH = -\frac{1}{2}Q_1 l_1 = -\frac{1}{2}Q_1^2/M$, where M is Young's elastic modulus. This can only be satisfied if Q is of the form λH^2, where λ is a constant, and it then gives $d\kappa/dQ = -2\lambda/M$, and the elongation l is $-\frac{1}{2}d\kappa/dQ\,.\,H^2$, that is $\frac{1}{2}\,d\kappa^{-1}/dQ\,.\,I^2$, or $-\frac{1}{2}\,d\log\kappa/dQ\,.\,HI$; while the corresponding stress Q is $-\frac{1}{2}d\kappa/dl\,.\,H^2$.[*] This result is of course valid only in the absence of hysteresis. A similar process applies where the field is transverse to the bar; and thus Kirchhoff's complete results may be obtained. A more complete enumeration of possible physical changes would also take cognizance of alteration of the elastic constants of the material due to the magnetic excitation; but this cause (*cf.* § 83) will not add terms of the first order to the energy-changes unless the bar is under extraneous stress, not merely constraint, while the cycle is being performed.

For dielectrics, direct experiments have not found any sensible dependence of inductive capacity on the pressure in the case of liquids; while the experimental discrepancies,[†] which these terms were introduced by von Helmholtz to reconcile, have since been cleared up.

68. In the paper above referred to, Kirchhoff remarks (§ 3) that an expression for the traction across an ideal interface in a uniform polarized medium might be arrived at by supposing a very thin film of air introduced along the interface, and computing the attraction between the two layers of opposed poles thus separated, a process which had been employed by Boltzmann. He concludes that this process must be at fault, on the ground that the specification of stress thus obtained does not satisfy a necessary property of mechanical stress-systems, namely that the tractions exerted over the surface of an infinitesimal element of volume of any form must balance each other; and he gives this as the reason for having to fall back on an energy-method in order to obtain a specification free from objection. The preceding considerations (§§ 44–48) indicate the direct reason of the illegitimacy of that process, while they also exhibit the logical basis of the application of the method of mechanical energy in problems of molecular physics.

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[*] [In these differentiations I is constant; see § 83.]

[†] Namely, the differences in the values of K at first found by Quincke, by use of three different experimental methods (§ 78), which it is easy to see would on the usual theory involve perpetual motions.

Conservation of Energy in the Electric Field: Limited Validity of POYNTING'S *Principle.*

69. It has been explained (§ 6) that the agencies in an electric field may be in part traced by transmission through the æther after the manner of ordinary mechanical stress, and in part, namely as regards forces on the electrons, not so traced. As regards the former part, therefore, the increase of energy in any region must be expressible explicitly as a surface integral, representing work done by tractions exerted over its boundary. This theorem will thus have application in all cases in which the configuration of the electrons is not changing; for its strict application, the bodies inside the region which carry currents or electric charges or are polarized, must thus be at rest, and there must be no change in the state of electrification of any conductor in the region. Recurring for an illustration to the simpler circumstances of a perfect fluid containing vortex rings, it is easy to show analytically that the rate of increase of energy in any region is there expressible as a surface integral, involving the velocity and the pressure, only when there are no vortex rings in the region or when the rings in it are all supposed to be held fixed by constraint.* This illustration also emphasizes the point that the surface integral must be taken as a whole, that an element of it does not necessarily represent the activity across the corresponding portion of the surface.

Thus taking the material system, concerning which we need make no hypothesis as regards inductive quality or æolotropy, to be *at rest* in the electric field so that there are no changes of energy due to the mechanical forcives, and neglecting those due to convection currents which rearrange electrifications, if W and T denote the organized potential and kinetic electric energies in the region, and D the rate of dissipation of organized energy due to currents of conduction, $d\mathrm{W}/dt + d\mathrm{T}/dt + \mathrm{D}$ must be expressible as a surface integral. Now W is made up of the energy of æthereal strain $(8\pi)^{-1}\int(\mathrm{P}^2 + \mathrm{Q}^2 + \mathrm{R}^2)\,d\tau$, and that of material polarization, $\int\phi\,d\tau$ where $\phi = \int(\mathrm{P}\,df' + \mathrm{Q}\,dg' + \mathrm{R}\,dh')$ which must be an exact differential when there is no dielectric hysteresis; thus in all $d\mathrm{W}/dt = \int(\mathrm{P}\,df''/dt + \mathrm{Q}\,dg''/dt + \mathrm{R}\,dh''/dt)\,d\tau$. Again the rate of dissipation arising from ionic migration in the conducting circuits is $\mathrm{D} = \int\{\mathrm{P}(u - df''/dt) + \mathrm{Q}(v - dg''/dt) + \mathrm{R}(w - dh''/dt)\}\,d\tau$. Hence we must have $d\mathrm{T}/dt = d\mathrm{E}/dt - \int(\mathrm{P}u + \mathrm{Q}v + \mathrm{R}w)\,d\tau$, in which $d\mathrm{E}/dt$ is a surface integral; and this equation will give an *a priori* indication, independent of special hypothesis, of the distribution of organized kinetic energy in the medium, that of the potential

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* The reason is simply that the form of the contained vortex rings is not a function merely of the state of the fluid inside the surface, but also in part determines the simultaneous velocity of the fluid throughout all space. So also the energy associated with each atom of matter is really distributed throughout the whole æther, and therefore the energy-changes associated with changes in the configuration of matter cannot be represented as propagated step by step across the æther.

energy and the dissipation being supposed known. Substituting from the kinematic relation $4\pi u = d\gamma/dy - d\beta/dz$, and integrating by parts,

$$dT/dt = dE/dt - (4\pi)^{-1}\int\{l(\beta R - \gamma Q) + m(\gamma P - \alpha R) + n(\alpha Q - \beta P)\}\,dS$$
$$+ (4\pi)^{-1}\int(\alpha\,da/dt + \beta\,db/dt + \gamma\,dc/dt)\,d\tau.$$

This equation of energy can however only apply to the case in which the energy of magnetic, as well as of electric, polarization is completely organized, and not mixed up with other molecular energy of the material, as it would be if there were hysteresis or permanent magnetism. When this condition is satisfied, the negation of perpetual motion requires that $\alpha\,da + \beta\,db + \gamma\,dc$ shall be an exact differential, say $d\psi$: thus we may tentatively assume $T = (4\pi)^{-1}\int\psi\,d\tau$, when the surface integral will remain as the value of dE/dt. In the case usually considered, in which the law of induced magnetization is linear, this gives MAXWELL'S formula for the distribution of the energy, $T = (8\pi)^{-1}\int(a\alpha + b\beta + c\gamma)\,d\tau$; while the value of dE/dt expresses POYNTING'S law of flux of electric energy corresponding to that hypothesis.

70. That this law of distribution of electrokinetic æthereal energy, for a magnetic medium of constant permeability, falls in with the present scheme may be verified as follows. Let $(\alpha', \beta', \gamma')$ be proportional to the velocity of the irrotational flow of the æther, due in part when there is magnetism to the AMPEREAN æthereal vortices, in such wise that the total kinetic energy is $(8\pi)^{-1}\int(\alpha'^2 + \beta'^2 + \gamma'^2)\,d\tau$; this is equal to $(8\pi)^{-1}\{\int\nu dV/dn\,dS - \Sigma\mathfrak{k}\int dV/dn\,d\sigma\}$, where $\delta\sigma$ is an element of a barrier surface closing a magnetic vortex of strength $\mathfrak{k}$, and δS is an element of the outer boundary of the region under consideration. As T is to include only the organized energy, it is given by this expression when in it V is restricted to be the potential of the magnetic force as ordinarily defined. In that case for an element of volume $\delta\tau$,

$$\Sigma\int\mathfrak{k}\,dV/dn\,d\sigma = \Sigma(\mathfrak{k}l\,dV/dx + \mathfrak{k}m\,dV/dy + \mathfrak{k}n\,dV/dz)\sigma = -4\pi(A\alpha + B\beta + C\gamma)\,d\tau;$$

and therefore

$$T = (8\pi)^{-1}\int(\alpha^2 + \beta^2 + \gamma^2)\,d\tau + \tfrac{1}{2}\int(A\alpha + B\beta + C\gamma)\,d\tau = (8\pi)^{-1}\int(a\alpha + b\beta + c\gamma)\,d\tau.$$

But although this expression locates the energy correctly as regards distribution throughout space, it still ignores the essential distinction between the energy of the translatory motions of electrons which constitute the current and that of their orbital motions which involve the magnetism; in a complete and consistent theory these two parts must be kept separate; *cf.* foot-note, § 38 *supra.*

On the Nature of Paramagnetism and Diamagnetism, as indicated by their Temperature Relations.

71. As a result of an extensive investigation of the magnetic properties of matter, the law has recently been formulated by CURIE* that in all feebly paramagnetic sub-

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* P. CURIE, 'Annales de Chimie,' 1895.

stances, including gases, the coefficient of magnetization varies inversely as the absolute temperature, with a degree of accuracy which tends to perfection at high temperatures: that in strongly magnetic substances such as iron, nickel, and magnetite, the same law is ultimately reached when the temperature is sufficiently high: while in diamagnetic substances the coefficient is usually nearly independent of temperature and also of changes in the chemical state of the material. The inference is made by CURIE that this points to diamagnetism being an affair of the internal constitution of the molecule, having only slight relation to the bodily motions of the molecules on which temperature depends; which is in accordance with the modified WEBERIAN view necessitated by the present theory. On the other hand, paramagnetization is an affair of orientation of the molecules in space without change of internal conformation, so that alteration of the mean state of translational motion is involved in it, and we should expect a temperature effect. A striking and probably just analogy is drawn by CURIE between (i) the simple law of expansion of a gaseous substance at high temperature, and the sudden change which it undergoes on lowering the temperature beyond a critical point so that the mutual attractions of the molecules come into play and produce the liquid state, and (ii) the simple law of magnetization of a substance like iron or nickel at high temperatures, and the sudden change which it undergoes when the temperature is lowered beyond the point at which the material passes into its strongly magnetic or ferromagnetic condition. The relation between paramagnetization and temperature in the former state proves to be so simple and universal that it must be the expression of a theoretical principle. The following considerations in fact derive it from CARNOT's principle: the argument is precise so long as the induced magnetization is so slight that the exciting magnetic force on the separate molecules is practically that of the inducing field, but it loses exactness as soon as, owing to diminution of energy of agitation with falling temperature, the molecules begin to exercise sensible magnetic control over each other, and thus introduce the phenomena of grouping and consequent hysteresis that are associated with the ferromagnetic state.

72. Consider a mass of paramagnetic material, moved up from a place where the intensity of the magnetic field vanishes to a place where it is H. The aggregate per unit volume of the total magnetic energies of its molecules is thereby altered from null to $-$ IH or $-\kappa H^2$. The mechanical work done by the mass in virtue of its attraction by the field is $\frac{1}{2}$IH, for the magnetization is at each stage of its progress proportional to the inducing force. Thus there remains a loss in the total magnetic energy of the molecules, equal to $\frac{1}{2}$IH; this can only have passed into heat in the material; for we can work on the hypothesis that the field of force H is due to an absolutely permanent magnetic system, so that no energy is used up in producing magnetic displacements in the inducing magnets. Now let us apply CARNOT's principle to a reversible cycle in which the material is moved up into the field at temperature $T + \delta T$ and removed at temperature T, with adiabatic transition

between these temperatures. Let $h + \delta h$ be the thermal energy per unit volume which it must receive from without at the higher temperature, and h that which it must return at the lower, in order to perform the amount of work δW, equal to $\frac{1}{2} H^2 d\kappa/dT.\delta T$, in the cycle; then, by CARNOT's principle, $\delta W/\delta T = h/T$; and $h = -\frac{1}{2}\kappa H^2$ as above; so that $d\kappa/dT = -\kappa/T$, leading to $\kappa = A/T$ which is CURIE's law. Conversely, assuming CURIE's law we can deduce that in paramagnetic bodies magnetization consists in orientation of the molecules without sensible change in their internal energies. In an analytical form the argument will then run as follows: $dh = MdI + NdT$, and $dE = dh - \kappa^{-1}IdI$; whence by the thermodynamic formula $M/T = d/dT.\kappa^{-1}I$, so that $M/I = Td\kappa^{-1}/dT, = \kappa^{-1}$ by CURIE's law; hence $dh = HdI + NdT$, so that at constant temperature $h = \frac{1}{2}HI$, that is the heat that the material developes during magnetization is the equivalent of the magnetic energy that is not used up in mechanical work. This is precisely what we should expect if the material is a gas; for there is then no internal work by which this energy could be used up, and the magnetization arises from the effort of the magnetic field to orientate the molecules which are spinning about as the result of the gaseous encounters. The law of CURIE thus indicates that the same is sensibly true for all paramagnetic media at high temperatures: at lower temperatures they gradually pass into the ferromagnetic condition. It is the magnetization, so to speak, of an ideal perfect ferromagnetic, in which the controlling force that resists the orientating action of the field is practically wholly derived from the magnetic interaction of the neighbouring molecules, which for this purpose form elastic systems, that is illustrated by EWING's well-known model, which so clearly represents the hysteresis accompanying ferromagnetic excitation. In ordinary paramagnetic substances this mutual magnetic control of the molecules is insensible compared with the control due to other molecular causes; and our conclusion is that these causes are such that the magnetic energy expended in working against them is transformed into heat energy, not into internal energy of any regular elastic type.

But we have not taken account of the fact that the molecules of every substance are subject to both paramagnetic and diamagnetic influence, of which one or the other preponderates. The theoretical law should thus be $\kappa = -B + AT^{-1}$ or $\kappa T = A - BT$; so that in a diagram of the relation between κT and T each substance would be represented by a straight line. In paramagnetics the line should slope slightly down towards the axis: for diamagnetics it should pass not through the zero of temperature but on the positive side of it. According to CURIE, his experimental results are equally well represented by this formula, on account of the preponderant influence of the paramagnetism.

Similarly, should it turn out that for weakly electric media such as gases, $(K-1)/\rho$ is independent of the temperature, it would follow that the electric polarization is mainly an affair of change of internal constitution of the molecules: while were the polarization mainly an affair of molecular orientation, $(K-1)/\rho$ would vary inversely as the absolute temperature: in intermediate cases it would not vary so rapidly as this. The circumstance that in gaseous media and some others, the dielectric constant is exactly equal to the square of the refractive index, favours the former alternative (§ 21).

Mechanical Relations of Radiation reconsidered.

73. The results given in Part II., §§ 28–9, as to the mechanical forcive exerted on a material medium by a stream of radiation passing across it, require amendment in the light of these principles, of which they also form an apt illustration. Consider, as there, two media separated by the plane of yz, and a system of plane-polarized waves advancing across them, with their fronts parallel to that plane, the electric vibration parallel to the axis of y, and the magnetic one parallel to the axis of z; we may generalize by taking K and μ to be in each medium functions of x. The electrical equations are

$$4\pi v = -\frac{d\gamma}{dx}, \qquad \frac{dQ}{dx} = -\frac{dc}{dt}, \qquad v = \sigma Q + \frac{K}{4\pi} C^{-2} \frac{dQ}{dt}.$$

Applying the formulæ found above (§ 38), the force acting on the electric polarity comes out to be null, while the electromagnetic bodily force is, per unit volume, $X = v'\gamma$, being wholly parallel to the axis of x: its periodic part has double the frequency of the radiation. Now there is no mechanical elasticity associated with matter which is powerful enough to transmit in any degree the alternating phases of forcives connected with a phenomenon which travels so fast and with such short wave-length as radiation, long HERTZIAN waves being excluded. Consequently when X is wholly alternating it is not transmitted by material stress at all; and it is only when its value for each element of the medium contains a non-alternating part that we can have a material forcive. When the media are perfectly transparent, and are traversed by a steady train of waves, there is therefore no transmissible material forcive either on surfaces of transition or anywhere else, and the $\int X\,dx$ previously calculated has no relation to material stress. But if we consider a stream of radiation passing across a transparent medium into an opaque one, and for simplicity take the latter to be homogeneous so that for the transmitted waves $c = c_0 e^{-px} \cos(nt - qx)$, the expression for X, viz: $\left(v - \frac{1}{4\pi} C^{-2} \frac{dQ}{dt}\right)\gamma$, contains a non-periodic term $\left(1 - \frac{\mu n^2 C^{-2}}{p^2 + q^2}\right) \frac{pc_0^2}{8\pi\mu^2} e^{-2px}$. This when integrated over the medium gives a pressure on the opaque medium of intensity $\left(1 - \frac{\mu n^2 C^{-2}}{p^2 + q^2}\right) \frac{CqE}{2\mu n}$, where E is the energy per unit volume of the incident radiation absorbed. Unless the opacity is so great that the intensity of the light is diminished in the ratio e^{-1} in penetrating a few wave-lengths, that is when p is negligible compared with q,* this pressure will be

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* This will not usually be the case for metallic media.

The sign of this mechanical force may be negative in a region of intense absorption.

practically $(1 - \mu m^{-2}) m\mathrm{E}/2\mu$, where m is the real part of the index of refraction of the medium, as measured by the ratio of the velocities in deviation experiments with prisms. And in general it appears that it is only absorption, not reflexion, of radiation that is accompanied by a mechanical forcive, the force on any absorbing mass being $(1 - \mu m^{-2}) m\mathrm{E}'/2\mu c$ in the direction of the radiation traversing it, where E′ is the total radiant energy absorbed by the mass per unit time.

In the case of a transparent medium traversed by two systems of waves, direct and reflected, forming stationary undulations, the mechanical force is proportional to $\sin 2qx$, and vanishes at both the nodes and antinodes of the electric vibration.

74. The *rationale* of the mechanical forcive is that, owing to the absorption of energy, which can only occur when phase-differences exist, a difference of phase becomes established between the two factors of X, the electric current and the magnetic field, so that their product contains a non-alternating part. It is known that vapours of complex chemical constitution are very powerful absorbers of radiation; and in this case (if not in all cases) the absorption must be a property of the single molecule. By the argument just stated, there must then be difference of phase between the electric flux (displacement of electrons) in the molecule and the magnetic field, and this will give a mechanical forcive driving the molecule along the path of the radiation. As the tails of comets and the Solar Corona consist of very rare distributions of vaporous or other matter, in free space which exerts no retarding influence, a comparatively small absolute amount of absorption by them of the Solar radiation might account for their observed repulsion from the Sun; in this way a definite and actually existing physical agency may be made to take the place of vague electrical repulsion in BREDICHIN'S important analysis of cometary appendages.

This mechanical action of waves on absorbing systems placed in their path may be roughly illustrated by an arrangement in which a system of sound waves traverses a space filled with resonators approximately in unison with them. The open mouth of each resonator is repelled,* so that in case there is any regularity in their orientation, the system as a whole will be subject to mechanical force. The resonators might be suspended so that the mechanical forces may themselves produce this orientation, and thus form a sort of medium polarizable by waves. A correponding electric illustration is the action of long HERTZIAN waves in orientating and repelling mobile conducting circuits which lie in their path. The very considerable repulsion of the vanes in the radiometer arises of course from a mutual stress between the vanes and walls and the rarefied gas, and so has a null resultant as regards the system as a whole.

Stresses and Deformations in Electric Condensers.

75. The elastic deformation produced in the dielectric of a spherical condenser by

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* *Cf.* Lord RAYLEIGH, 'Theory of Sound,' vol. 2, §§ 255*a*, 319.

the mechanical forcive may be readily calculated. If u denote the radial displacement, the normal and transverse principal tractions at any point in the spherical dielectric shell are

$$\mathrm{P} = \lambda\left(\frac{du}{dr} + 2\frac{u}{r}\right) + 2\mu\frac{du}{dr}, \quad \mathrm{Q} = \lambda\left(\frac{du}{dr} + 2\frac{u}{r}\right) + 2\mu\frac{u}{r},$$

where μ, $\lambda + \frac{2}{3}\mu$ are the moduli of rigidity and compressibility of the material. The electric force at any point is kr^{-2}, where $k\mathrm{K}$ is the charge on a coating: hence (§ 36) the mechanical bodily force is derived from the potential $-(\mathrm{K}-1)/8\pi \,.\, k^2r^{-4}$; and there is also an outward normal traction over each coating equal to $-\mathrm{K}/8\pi \,.\, k^2r^{-4}$. The equation of equilibrium of a conical element of volume is

$$\frac{d}{dr}(\mathrm{P}r^2) - 2\mathrm{Q}r = \frac{\mathrm{K}-1}{2\pi}\frac{k^2}{r^3}, \text{ so that } \frac{d}{dr}\left(r^2\frac{du}{dr}\right) - 2u = \frac{\mathrm{K}-1}{2\pi(\lambda+2\mu)}\frac{k^2}{r^3},$$

giving

$$u = \mathrm{A}r + \frac{\mathrm{B}}{r^2} + \frac{\mathrm{K}-1}{\lambda+2\mu}\frac{k^2}{8\pi r^3},$$

and therefore

$$\mathrm{P} = (3\lambda + 2\mu)\mathrm{A} - \frac{4\mu}{r^3}\mathrm{B} - \frac{\lambda+6\mu}{\lambda+2\mu}\frac{(\mathrm{K}-1)k^2}{8\pi r^4}, \quad \mathrm{Q} = (3\lambda+2\mu)\mathrm{A} + \frac{2\mu}{r^3}\mathrm{B} - \frac{\lambda-2\mu}{\lambda+2\mu}\frac{(\mathrm{K}-1)k^2}{8\pi r^4}.$$

The values of A and B are determined by the normal tractions at the coatings $r = r_0$ and $r = r_1$, and are, when the coatings are wholly supported by the dielectric,

$$\mathrm{A} = \frac{1}{r_1r_2(r_1^2 + r_1r_2 + r_2^2)}\frac{\vartheta k^2}{8\pi(3\lambda+2\mu)}, \quad \mathrm{B} = \frac{r_1^3 + r_1^2r_2 + r_1r_2^2 + r_2^3}{r_1r_2(r_1^2 + r_1r_2 + r_2^2)}\frac{\vartheta k^2}{32\pi\mu},$$

where ϑ represents $1 - (\mathrm{K}-1) \,.\, 4\mu/(\lambda + 2\mu)$.

It will suffice to state the results for the case of a thin shell of radius a with adhering coatings: then $\mathrm{A} = \vartheta k^2/24\pi(3\lambda + 2\mu)a^4$, $\mathrm{B} = \vartheta k^2/24\pi\mu a$.

The coefficient of expansion of the radius of the sphere, due to the electric stress, is $u/a = (\lambda + 2\mu - \mu\mathrm{K})/8\pi\mu(3\lambda + 2\mu) \,.\, \mathrm{F}^2$, and the coefficient of expansion of the volume of the sphere is three times this. It is easily verified that when the shell is thin the stress in the material of the dielectric is made up of a pressure $\mathrm{KF}^2/8\pi$ normal to the shell combined with a pressure $(\mathrm{K}-2)\mathrm{F}^2/8\pi$ in all directions tangential to it.*

[76†. The circumstance that these results are independent of the radius of the sphere suggests an extension of their scope. Whatever be the form of the dielectric shell provided it is of uniform thickness, F will be the same all over it; and the mechanical force acting on its substance, being derived from a potential $-(\mathrm{K}-1)\mathrm{F}^2/8\pi$, will be directed at each point along the normal δn to the shell. Consider the internal equilibrium of an element of volume $\delta\mathrm{S}\,\delta n$, of which the opposite faces $\delta\mathrm{S}$ are elements of level surfaces bounded by lines of curvature for which R_1, R_2 are the principal radii: it will be maintained if a pressure of intensity P,

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* This naturally differs from KIRCHHOFF'S result, 'Wied. Ann.' 24, p. 52, § 4.

† Rewritten December 2.

equal to $\mathrm{KF}^2/8\pi$, act on the element across the faces $\delta\mathrm{S}$, and another pressure $-\mathrm{Q}$ act on it, which is the same across all perpendicular faces. For, resolving the forces along δn, we must have for equilibrium

$$-\delta n \frac{d}{dn}(\mathrm{P}\,\delta\mathrm{S}) - \mathrm{Q}\,\delta n\,\delta\mathrm{S}\left(\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right) + \frac{\mathrm{K}-1}{4\pi}\mathrm{F}\frac{d\mathrm{F}}{dn}\delta\mathrm{S}\,\delta n = 0.$$

Now, by the constancy of the induction, we have $d/dn\,(\mathrm{F}\,\delta\mathrm{S}) = 0$, leading to $\frac{d\mathrm{F}}{dn} = -\mathrm{F}\left(\frac{1}{\mathrm{R}_1} + \frac{1}{\mathrm{R}_2}\right)$ and also to $\frac{d}{dn}(\mathrm{P}\,\delta\mathrm{S}) = \frac{\mathrm{K}}{8\pi}\delta\mathrm{SF}\frac{d\mathrm{F}}{dn}$: thus, on substitution, we obtain $-\mathrm{Q} = (\mathrm{K}-2)\,\mathrm{F}^2/8\pi$. The constancy of Q all round the edge of a flat element of volume $\delta\mathrm{S}\,\delta n$ secures the balancing of the tangential components of the forces. Hence the mechanical stress in any condenser sheet of uniform thickness is the same as has been found above for the spherical case. If e and f denote the elongations of the material in the normal and tangential directions, $\lambda(e+2f) + 2\mu e = -\mathrm{KF}^2/8\pi$, $\lambda(e+2f) + 2\mu f = -(\mathrm{K}-2)\,\mathrm{F}^2/8\pi$; hence $f = (2 + \lambda/\mu - \mathrm{K})/(3\lambda + 2\mu)\,.\,\mathrm{F}^2/8\pi$, $e = (2\lambda/\mu - \mathrm{K})/(3\lambda + 2\mu)\,.\,\mathrm{F}^2/8\pi$; so that the extension of the volume of the shell and the change in its thickness are the same as were found above for the spherical case. If the shell is an open one, the presence of its free edge will disturb these relations: but that influence will be mainly local, as the forces introduced by the edge will be almost wholly of the nature of local action and reaction.]

Various Practical Illustrations and Applications of the Stress Theory.

77. *Refraction of a Uniform Field of Electric Force.*—An arrangement by which these principles may be precisely verified is that of the refraction, at a plane interface AB, of a sheaf of parallel lines of electric force F, according to the FARADAY-MAXWELL law of tangents, $\tan\iota_1/\tan\iota_2 = \mathrm{K}_1/\mathrm{K}_2$, $\mathrm{F}_1/\mathrm{F}_2 = \operatorname{cosec}\iota_1/\operatorname{cosec}\iota_2$. This configuration of lines of force may be obtained and fixed by means of a condensing system having its plates $\mathrm{P}_1\mathrm{Q}_1$ and $\mathrm{P}_2\mathrm{Q}_2$ normal to the incident and refracted lines. Each plate may be protected from convective discharge into the fluid dielectric by a covering plate of glass or mica, which will itself produce no refraction. In such a case, when both the dielectric media are fluid, the total mechanical result of the electric excitation will be the same as that of normal tractions on the interface between them, of intensity $-2\pi n'^2 - \frac{1}{2}i'\mathrm{F}$ towards each side, that is, in all $(\mathrm{K}_2 - \mathrm{K}_1)\,(2\pi n''^2/\mathrm{K}_1\mathrm{K}_2 - \mathrm{T}^2/8\pi)$ towards the side 1. As the field of force in this condensing system is uniform except near its edge, the interface will simply be lifted up between the plates by the amount which corresponds to this traction, without ceasing to be horizontal. Thus the common surface will be elevated when $\tan\iota_1$ is less than $(\mathrm{K}_1/\mathrm{K}_2)^{\frac{1}{2}}$, while at greater incidences it will be depressed.

This principle supplies in fact a method of obtaining the inductive capacities of fluid media by angular measurement only, without the aid of an electrometer. When the condenser $\mathrm{P}_1\mathrm{Q}_1$, $\mathrm{P}_2\mathrm{Q}_2$ is charged, the interface between the fluids will usually cease to be horizontal; the upper plate P_1Q_1 is then to be rotated until horizontality is again obtained, as may be tested very exactly by reflexion of a beam of light; then the ratio of the tangents of the inclinations of the plates will be that of the inductive capacities of the media. The method would also apply to solids, as we might employ a prism of the material, over a horizontal face of which a sheet of a fluid dielectric is spread, and observe the deviations from horizontality of the upper surface of this sheet. An equivalent arrangement has been actually employed for solids by PÉROT,[*] who however adjusted the plates to uniformity of the electric field by the electric test that translation of a small piece of solid dielectric in the field between them should not affect the capacity of the condenser.

78. *Experiments on Electric Traction and Change of Pressure in Fluids.*—The direct experimental examination of the material forcives of polarized media is necessarily confined to fluids, for in the case of solids the strains produced by them could hardly be disentangled from the intrinsic changes of configuration due directly to the polarization. The field for fluids has been very fully explored by QUINCKE.[†] The inductive capacity of the fluid dielectric of a horizontal condenser was first determined by direct electrical measurement. The attraction between the plates was then weighed. Then, using a wide cylindrical air-bubble extending across the space between the plates, and connected through an aperture in the upper one with a manometer, the increase of air-pressure in the bubble produced by charging the condenser was measured. As half-way between the plates the capillary interface between air and liquid lies along the lines of force, there is by the previous formulæ (§ 37) no true surface traction on that part of the interface; so that the indication of the manometer would give exactly the change of pressure in the liquid due to the electric excitation, were it not that the different electric conditions over other parts of the interface change the value of its curvature and so introduce a capillary change of pressure. Finally, employing a flat bubble of air resting against the upper plate alone, and maintaining the pressure in it constant, the change of curvature of its lowest part produced by the electrical excitation was measured by the optical method; the surface tension operating through this change of curvature must balance exactly the direct traction on the surface and the change of pressure in the liquid below it. To compute these, we notice that the line of force through the middle of the bubble

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[*] PÉROT, 'Comptes Rendus,' 1891; quoted by DRUDE, 'Physik des Æthers,' p. 299. The conjugate condensing system, in which namely the lines of force and the lines of equal potential are interchanged, so that the plates are now bent according to the law of tangents where they cross the surface of the liquid, has recently been brought into requisition by PELLAT ('Annales de Chimie,' 1895), in order to derive the law of the traction on a dielectric interface from the expression for the energy of the dielectric system. As in these cases the field of force is uniform in both media, the bodily part of the mechanical force vanishes, and the interfacial traction proper thus constitutes the whole forcive: but that would not generally be so.

[†] G. QUINCKE, 'Wied. Ann.,' 19, 1883; or an abstract in the paper below cited, 'Roy. Soc. Proc.,' 52, 1892, pp. 59–62.

is straight, so that if F denote the electric force in air and F/K that in the liquid, the traction on the surface due to these two causes is $(K-1)^2 F^2/8\pi K^2 + (K-1) F^2/8\pi K^2$, that is $(K-1) F^2/8\pi K$ upwards, which is the formula employed by QUINCKE; while F is determined by the relation $aF + bF/K = V$, provided the bubble is so broad that the middle tube of induction is practically cylindrical, a and b being the lengths of this tube that are in air and the liquid, and V the difference of potential between the plates. After an error in the direct determination of K, due to an experimental oversight whose existence was suggested by HOPKINSON, had been corrected, all the results showed substantial agreement, for thirteen liquids that were examined, with these theoretical formulæ. But the agreement was not quite complete; a subsequent examination* still showed that the attraction between the plates always came out less and the change of pressure in the liquids greater than the formulæ would give, though these discrepancies were within the limits of experimental error, except for the case of rape oil in which they amounted to as much as ten per cent. The neglect of the capillary correction above mentioned would account for a discrepancy in the same direction as the first; and the irregularity of the electric distribution near the edge of the plates would account for one in the same direction as the second.

In a paper on the bearing of the phenomena of electric stress on electrodynamic theory,† I had previously been led to inferences militating against the possibility of dielectric polarization being of molecular type, from a comparison of these experimental results with an electric traction formula including both the molar and the molecular forcives. According to the present argument (§ 44), the latter forcive being separately compensated, the difficulty there encountered does not exist. The remaining considerations in that paper retain their validity; they show for instance that the formulæ for the experimental reductions can be derived from a knowledge of the distribution of the organized energy alone. But in the light of the present views, we are no longer restricted or even allowed to consider the induction in a dielectric as all of one kind; the total circuital induction is in fact made up of a material polarity combined with an æthereal elastic displacement, giving an apparent but natural complexity which it had previously been an aim to evade.

79. *Experiments on Electric Expansion in Solids and Fluids.*—The results obtained in § 76 may be applied to the discussion of a very thorough series of experiments on electric expansion, made by QUINCKE,‡ which appear hitherto not to have been correctly interpreted. Following the early experiments of FONTANA, and more recent ones by GOVI and DUTER, a condenser of the form of a glass thermometer-bulb was used, and the expansion of volume arising from electric excitation was read directly on its tube. By employing a long cylindrical bulb, the

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* G. QUINCKE; 'Wied. Ann.,' 32, 1887, p. 537.

† 'Roy. Soc. Proc.,' 52, 1892, pp. 65–6.

‡ Abstracted in 'Sitz. Akad. Berlin,' February, 1880, and 'Phil. Mag.,' July, 1880, pp. 30–39: in full in 'Wied. Ann.,' 10, 1880, pp. 161–202, and 513–553.

longitudinal expansion of the glass could also be measured microscopically at the same time. It was found by QUINCKE that the coefficient of volume expansion was always about three times that of this longitudinal expansion of the glass dielectric, just as the above theory indicates for the elastic strain in a bulb of uniform thickness. The erroneous deduction was however made from an imperfect theory, that electric expansion of solids is uniform in *all* directions, like expansion by heat, and so in no part due to mechanical forces of attraction.

By using the formula of § 76 along with various known physical constants, a test of the order of magnitude of QUINCKE'S determinations may be obtained. Thus with a striking distance of ·4 centim. between brass balls 2 centims. in diameter, the expansion in volume of a flint glass condenser of thickness ·06 centim. was found to be $\frac{3}{4} \times 10^{-6}$.[*] According to BAILLE'S experiments[†] this striking distance corresponds to a difference of potential of 47 c.g.s. The expansion of volume of the bulb due to the mechanical force is $\{2 - K + \lambda/\mu\}/(\lambda + \frac{2}{3}\mu) . F^2/8\pi$, where, according to EVERETT'S experiments[‡] for flint glass, $\mu = 25 \times 10^{10}$ c.g.s. and $\lambda = \mu$, and according to HOPKINSON K is about 7. This gives for the thickness under consideration a coefficient of expansion equal to $-0{\cdot}24 \times 10^{-6}$, while the observed value was $0{\cdot}75 \times 10^{-6}$.[§] The difference between them, in so far as it does not arise from experimental uncertainties, is an intrinsic superficial expansion of the glass, which arises directly from the transverse polarization itself, and is not due to the mechanical forces caused by it. That there is such intrinsic alteration due to electric excitation, is independently suggested by QUINCKE'S observation that the values of the elastic constants of the material are slightly altered by that cause.

In the case of a fluid the effect of electric excitation is to diminish the hydrostatic pressure; consequently expansion should result when the plates of the condenser are fixed. This agrees with what happens for most fluids. But the fatty oils form an exception; thus for them at any rate there is an intrinsic electric contraction superposed on the expansion due to diminished pressure.

In both these cases the intrinsic change of volume is of order of magnitude not higher than the change due to the mechanical stress.

The observation of QUINCKE[||] that a thin glass-tube condenser, with walls thicker on one side than the other, becomes curved (in accordance with the theory above) when it is electrically charged, virtually affords a convenient method for studying

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[*] *Loc. cit.* 'Wied. Ann.,' 10, p. 190.

[†] 'Annales de Chimie,' 1882: quoted in J. J. THOMSON'S 'Recent Researches,' p. 87.

[‡] 'Phil. Trans.,' 1868, p. 369.

[§] According to QUINCKE'S own determinations (*loc. cit.* p. 187) the value of K would be about 12, which is a great deal higher than HOPKINSON'S results, and would give an expansion $-0{\cdot}30 \times 10^{-6}$. It seems possible that these determinations, which involve considerable unitary complexity, may be wrongly recorded, as QUINCKE'S reduction of them sometimes appears to give values for K that are less than unity.

[||] *Loc. cit.*, p. 394.

the gradual rise of the charge of the condenser and its residual discharge. With such a "glass thread electrometer" it appears that the curvature takes place gradually on excitation, occupying for small charges sometimes as long as 30 seconds: and on discharge it is annulled with corresponding slowness. As part of this deformation is intrinsic, that is, due to molecular forces and not to the mechanical stress, it is a direct indication of gradual shaking down of the material into modified molecular groupings under the influence of the electric field.

80. *Influence of Polarization of a Fluid on Surface Ripples.*—The last illustrations belong to cases in which the field of force is uniform, so that the bodily mechanical forcive vanishes. A problem amenable to experimental examination, in which this is not the case, is the influence of electric polarization on ripple motion in fluids. The fluid, in a glass dish, might for example form part of the dielectric of a horizontal condenser, of which the upper coating is a wire grating separated from the fluid by a plate of glass or mica so as to prevent communication of electrification to its surface.

Taking the axis of y downwards into the fluid whose dielectric constant is K_2, and the axis of x along the interface, the electric potentials in the two fluids, of which the upper will usually be air, are $V_1 = A_1 y + B_1 e^{my} \cos mx$, $V_2 = A_2 y + B_2 e^{-my} \cos mx$, subject to the condition that at the interface $y = C \cos mx$ we must have $V_1 = V_2$ and $K_1\, dV_1/dn = K_2\, dV_2/dn$. Thus $A_1 C + B_1 = A_2 C + B_2$, and $K_1 (A_1 + mB_1 \cos mx) = K_2 (A_2 - mB_2 \cos mx)$ which latter involves both $K_1 A_1 = K_2 A_2$ and $K_1 B_1 = - K_2 B_2$. The velocity potentials of the wave-motions in the two fluids are

$$\phi_1 = m^{-1}\, dC/dt\, e^{my} \cos mx,\ \phi_2 = - m^{-1}\, dC/dt\, e^{-my} \cos mx.$$

In addition to the hydrodynamical pressure-difference acting from the upper to the lower side of the interface, equal to $- g (\rho_2 - \rho_1) y + (\rho_2\, d\phi_2/dt - \rho_1\, d\phi_1/dt)$, that is to $- \{g (\rho_2 - \rho_1) C + m^{-1} (\rho_2 + \rho_1) d^2C/dt^2\} \cos mx$, there will act on it a downward capillary traction $T\, d^2y/dx^2$, or $- m^2 TC \cos mx$, and a downward electric normal traction* $(K_1 - K_2) (K_1 K_2^{-1} N_1^2 - T_1^2)/8\pi$, in which $N_1 = A_1 + mB_1 \cos mx$ while T_1 is of the second order. This electric traction is therefore equal to $K_1 (K_1 - K_2)/8\pi K_2 . (A_1^2 + 2mA_1 B_1 \cos mx)$, where $B_1 = (K_1 - K_2)/(K_1 + K_2) . CA_1$; while the intensity of the total displacement, material and æthereal, in the electric field is $i'' = - K_1 A_1/4\pi$. The balancing of these tractions at the interface $y = C \cos mx$ requires that, in addition to the mean statical elevation, we should have

$$(\rho_2 + \rho_1) \frac{n^2}{m^2} = (\rho_2 - \rho_1) \frac{g}{m} + mT - \frac{(K_2 - K_1)^2}{K_1 K_2 (K_2 + K_1)}\, 2\pi i''^2,$$

in which n/m is the velocity of wave-propagation. The effect of the electric polarization is thus for ripples of length $\lambda\ (= 2\pi/m)$ the same as would be that of a diminution of the surface tension by $(K_2 - K_1)^2/K_1 K_2 (K_2 + K_1) . \lambda i''^2$.

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* This is the statical equivalent as above (§ 37) of both the actual electric traction on the surface, and the electric pressure transmitted from the interior of the fluid to the surface.

If the lower medium were conducting, we should have had $A_1C + B_1 = 0$, and the electric downward traction would be $- K_1N_1^2/8\pi$, that is $- K_1A_1^2/8\pi + m/4\pi . A_1^2C \cos mx$. Thus in the equation giving as above the velocity of propagation, the electric term would be $- A_1^2/4\pi m$, or $- 4\pi\sigma^2/K_1^2m$, where σ is the density of the electrification on the interface. The effect of this electrification is thus the same as that of a diminution of the surface tension by $2\sigma^2\lambda/K_1^2$, where λ is the wave-length.*

81. *Relations of Electrification to Vapour Tension and Fluid Equilibrium.*—It has already been shown (§ 52) that the possibility of mechanical equilibrium between fluid dielectrics which do not mix requires that the electric tractions on the interface shall be in the direction of the normal. There are also other dynamical relations deducible from the fact that such forcives, when integrated round a closed circuit in fluid media, must give a null result, in order to avoid the establishment of cyclic perpetual motions. The earliest example which led the way to relations of this kind was Lord KELVIN's establishment of a connexion between the vapour tension of a liquid and the curvature of its free surface : and similar balances must independently hold good between vapour tension and other causes of surface traction.

Consider in the first place a volume of conducting fluid with a large horizontal free surface. Let an electric field be established over a portion of this surface ; there will be a surface density σ of electrification induced over that portion, which will vary from point to point ; while the electric forces will elevate the surface by an amount $h, = 2\pi\sigma^2/g\rho$ where ρ is the density of the fluid, above the level at a distance where there is no electrification. The vapour tension over the electrified part must thus be smaller by $g\rho_0h$ than over the unelectrified part, where ρ_0 is the density of the vapour. This difference of tension must be the natural steady difference produced by the electrification of the surface ; for otherwise a process of distillation will set in and there could not be equilibrium, though there could theoretically be perpetual generation of work while the temperature remains uniform, as the electric charge does not evaporate with the fluid. It follows that an electrification of surface density σ must depress the equilibrium vapour tension by an amount $2\pi\sigma^2\rho_0/\rho$.†

Suppose again that the fluid is a dielectric of inductive capacity K, and has no free charge. A similar train of reasoning shows, by the formula of § 37, that when the polarization of the material dielectric, at the surface, is made up of a normal component n' and a tangential component t', its vapour tension is thereby diminished by an amount $2\pi(Kn'^2 + t'^2)/(K - 1) . \rho_0/\rho$. Conversely, we can argue that, as the change of vapour tension can depend only on the state of polarization or electrification at the part of the surface which is under consideration, the effect of the electric excitation must be completely expressible by a mechanical traction over the surface

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* This result was given twice too large in 'Proc. Camb. Phil. Soc.,' April, 1890.

† This agrees with a result given by Prof. J. J. THOMSON, "Applications of Dynamics to Physics and Chemistry," 1888, § 86.

which must be wholly normal and depend only on the intensity of the field of force at the place. That this is the case for fluids, but not for solids, has already been shown. And this law of dependence of vapour tension on electric state only applies to fluids, not to solids like ice; for a flow of the medium is required to complete the cycle on which the argument is based. In the case of a solid with finite vapour-tension, electric excitation—as also gravity, strains, and other physical agencies—will promote evaporation, excessively slow of course, from some parts of its surface, and condensation on others, until a form suitable to equilibrium of vapour tension is attained.

In expressing conditions of equilibrium for fluid media, the above total electric normal traction over each interface is simply to be added to such other forces as would exist in the material system if there were no electric field. Thus if we take for example the case of a number of dielectric fluids superposed on each other in a tall jar under the action of gravity, the form of the upper surface is obtained by equating the electric traction to the pressure difference produced by difference of level alone; and for any interior interface the same statement holds good, the form of each interface depending only on the electric field at the place and the inductive capacities of the two fluids which it separates. And this procedure is quite general whatever extraneous forcives there may be; the form of each interface is always determined by equating the difference of electric tractions on its two sides to the difference of pressures due to other than electric causes.

82. *Tractions on the Interfaces of a divided Magnetic Circuit.*—An important practical deduction is that when a bar or ring, longitudinally magnetized temporarily or permanently, is divided by an air-gap, the force drawing together the two halves of it consists of the attractions of the uncompensated polarities which would remain if there were no air-gap, together with a traction on each face of the gap, at right angles to its plane, and of intensity $2\pi\nu^2$, where ν is the normal component of the magnetization. This traction is in other respects quite independent of the character of the magnetic field that may exist at the gap; when the gap is narrow it is simply the attraction between the free polarities on its two faces. When the gap is transverse, the total amount of the traction is $2\pi\int I^2\, dS$, that is $(8\pi)^{-1}\int(B - H)^2\, dS$, where B and H are the longitudinal components of the magnetic induction and force; when it is oblique the longitudinal pull between the halves of the bar varies as the square of the cosine of the obliquity. When the substance is magnetized by an electric coil, there may in addition be the attraction between the two halves of the coil. For the case of iron H is very small compared with B, unless the field is far greater than is required to saturate the iron; so that the part of the mechanical traction across a transverse gap which is due to the polarities on its faces is practically $(8\pi)^{-1}\int B^2\, dS$.

83. *Interaction of Mechanical Stress and Magnetization.*—Consider a wire, magnetized to intensity I by a longitudinal magnetic field H, and subject to an

extraneous tensile force of intensity Q per unit cross-section: let M denote the modulus of elastic extension of its material, which will be an even function of I. The mechanical work expended on the wire in a slight alteration of its circumstances is per unit volume

$$\delta W = (M^{-1}\,\delta Q + \delta\eta)\,Q + H\,\delta I = (M^{-1}Q + Q\,d\eta/dQ)\,\delta Q + (H + Q\,d\eta/dI)\,\delta I,$$

where η is its intrinsic magnetic elongation when magnetized to intensity I under tension Q, this magnetization practically not altering the extraneous field in the case of a wire. To avoid perpetual motions, δW must in the absence of hysteresis be a perfect differential of the independent variables I and Q: hence $Q\,dM^{-1}/dI = (dH/dQ)_I + d\eta/dI$. Here I is a function of H and Q, so that to determine dH/dQ when I is constant we have $(dI/dH)_Q + (dI/dQ)_H\,(dQ/dH)_I = 0$, the subscript denoting the variable that is constant in the differentiation. Thus on substitution $d\eta/dI = (dI/dQ)_H/(dI/dH)_Q + Q\,dM^{-1}/dI$; and the total expansion is $\eta' = \eta + Q/M$: so that on writing as usual κ for $(dI/dH)_Q$, we have $d\eta'/dI = -I\,(d\kappa^{-1}/dQ)_H + 2Q\,dM^{-1}/dI$. This is the exact equation which should be directly satisfied by series of observations of η', I, and M, formed with different constant values of H and Q, provided hysteresis is negligible. As η' must be an even function of I, it follows that when I and Q are small, $\eta' = -\frac{1}{2}I^2\,(d\kappa^{-1}/dQ)_H$, or $\frac{1}{2}I^2\,(d\kappa^{-1}/dQ)_I$, as in § 67.

For the case of a ring magnetized by a coil, there can be no free polarity except at an air-gap; thus there is no stress of magnetic origin in the material. The alterations of longitudinal and transverse dimensions of rings of iron and nickel* are thus wholly intrinsic changes due to the magnetic polarity and in no part due to mechanical stress such as Q. In the neighbourhood of the origin, where η' is proportional to I^2, the curves given by BIDWELL expressing the relation between η' and I should be parabolic, as in fact they are. At the magnetization corresponding to a maximum or minimum ordinate η' of the curve, the effect of a very small imposed tension on the magnetization should change sign, being null for that particular magnetization; the summit of the curve is therefore the VILLARI critical point. But if there is a tension Q so considerable that change of the elastic modulus by magnetization contributes appreciably to the elongation, the VILLARI point will be displaced from the summit of the curve, backwards when magnetization increases the modulus. It appears from the experiments of BIDWELL† that for iron tension increases the intrinsic elongation, for nickel it at first increases then diminishes and finally for stronger fields increases it, while for cobalt there is no sensible effect.

84. *Mechanical Stress in a Polarized Solid Sphere.*—The mechanical stress sustained by a sphere of soft iron situated in a uniform magnetic field H can be simply expressed. The well-known analysis of POISSON gives a constant field

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* SHELFORD BIDWELL, 'Phil. Trans.,' A, 1888, p. 228; 'Roy. Soc. Proc., 1894.

† 'Roy. Soc. Proc.,' vol. 47, 1890, p. 480.

$H' = 3H/(\mu + 2)$ and uniform magnetization $I' = (\mu - 1)H'/4\pi$ inside the iron, whether the law of induction is linear or not. Thus for this case of a sphere the mechanical forces exerted on the iron involve no distribution of forcive throughout its volume, but simply an outward normal traction of intensity $\{(\mu^2 - 1)\sin^2\theta - \mu + 1\} H'^2/8\pi$ over its surface: that being so, the stresses agree with KIRCHHOFF's values, and the elastic strain produced in the sphere is given by his formulæ,[*] the result of course involving only very slight deformation. In fact, taking the axis of x along the direction of I', it is clear that an elastic displacement (u, v, w) of the type $u = ax^3 + bx(y^2 + z^2) + cx$, $v = w = a'x^3 + b'x(y^2 + z^2) + c'x$ satisfies the conditions of the problem for the case of a sphere, the constants being determined by satisfying the equations of internal equilibrium and adjusting the surface tractions. In addition to this mechanical deformation there will be the intrinsic deformation above determined (§ 83) arising from the molecular changes produced by the magnetic polarization.

Precisely similar formulæ express the mechanical stress in a sphere of solid dielectric matter situated in a uniform electric field.

I desire to express, as in previous Memoirs, my obligation to the friendly criticism of Professor G. F. FITZGERALD, which has enabled me to remove obscurities and in various places to make my meaning clearer.

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[*] KIRCHHOFF, "Gesamm. Abhandl., Nachtrag," p. 124; *cf.* also LOVE, "Treatise on Elasticity," I., § 168.

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ERRATUM.

P. 226, line 16, *for* $Ar\cos\theta$ *read* $Ar^2(3\cos^2\theta - 1)$, *and for* $Br^{-2}\cos\theta$ *read* $Br^{-3}(3\cos^2\theta - 1)$, *and replace the remainder of the sentence by the words* where the values of A, B, A′, B are determined by the surface conditions.